Exploring Math from Algebra to Calculus with Derive, A Mathematical Assistant for your Personal Computer

By Jerry Glynn

MathWare
604 E. Mumford Dr.
Urbana, IL 61801

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I have many people to thank for help in creating this book; I can mention only some of them here.

W. W. Sawyer's books about mathematics and teaching have been a major influence on me since I began teaching math. His clarity of thought and ability to communicate are an ideal I can only strive towards but will never reach. W.W.'s willingness to spend time and share his ideas with me during my first visit to England helped me to see what was possible.

Eva and John Gray introduced me to the new world of symbolic algebra via Mathematica and restarted me on my previous symbolic experience via Mu Math which led me to Derive.

Stephen Wolfram shared with me his vision of a world with proper computer tools to do math and then worked very hard to create such a tool and to distribute it showing us all what was needed.

Theo Gray answered my many beginner's questions while writing the front end of the Macintosh version of Mathematica and continues to be a great friend and help when I get in trouble (which is often).

I thank Bob Davis who has encouraged my teaching for years and who has been suggesting I write a book for all of that time.

I thank Robert Drayer for encouragement at a crucial time in the development of this book.

Joyce Glynn and David Eisenman have my thanks for editing and encouragement throughout the project and Joyce has been a major force in taking MathWare, the publisher of this book, from an idea to a reality.

My partner in The Math Program, Don Cohen, showed me what was possible by publishing his fabulous book, Calculus By and For Young-People (ages 7, yes 7 and up). Our successful partnership in The Math Program for the past 16 years supports our efforts to communicate what Don and I both believe about teaching math.

Mark Deininger has manipulated PageMaker and laser printers and graphics drivers and Derive and fonts and managed to create and carry through the design for this book which is far beyond what I hoped for at the beginning.

Foreword

The authors of Derive, David Stoutemyer and Al Rich, had a direct effect on this book at many levels. By creating Derive they gave me a tool which was both powerful and practical. I wanted to write a book for this program because I knew it could get out to the people who don't have big financial backing and who must work with modest every-day machinery. Al and David also are responsive to my suggestions for changes in Derive and this made me feel a part of the project and not just a user. Revision of this book benefited from the use of Derive, version 2.51.

The main inspiration in all of my teaching, over the past thirty-two years, has been my students. My own children David, Brian, Clair and Erin, have always amazed me with their mathematical abilities. Students like Laura Kate who was almost 5 when we began working, or Carolyn who has attended The Math Program from the first grade through the fifth (and does amazing mathematical things every week), or Roger who decided to do his first calculus class in the summer (against my recommendation) and then worked so hard that he made it a success, have encouraged me to complete this book because I saw what was possible in their work. To them and to my future students I offer this book for their consideration.

Jerry Glynn
Urbana, Illinois
July 27th, 1992

P.S. The Derive computer bulletin board has been going strong for the past three years. It is a joint venture between Soft Warehouse and MathWare. Greg Smith's bbs software has performed admirably. Address mail for Greg to: 917 W. Columbia Ave., Champaign, IL 61821.

The bulletin board is based in Urbana, IL at (217) 337-0926. It can handle communications speeds up to 14400 baud; settings, n (no parity), 8 (data bits), 1 (stop bit).

We hope that Derive users will call with questions, suggestions and good ideas to share and that potential users will make contact to learn about Derive, A Mathematical Assistant for Your Personal Computer.

Table of Contents

Chapter 1

Why We Really Like Derive

Why We Really Like Derive

My partner Don Cohen and I have used Derive for more than four years; our students have used Derive for almost the same length of time. Our students are between the ages of five and forty-five. Their experiences (and thus ours) are varied and complex. One of our students is a working engineer who is continuing to study calculus; others are unsuccessful first year algebra students. Some of our youngest students are learning to count by nines; others of our students are fifth graders trying to remember 7*8 = 56. Some of our elementary age students are studying graphs of trig functions, the binomial theorem, and calculus.

We have found that all of our students can benefit from using Derive, although often for different reasons. We will tell some of our experiences at The Math Program and show some typical problems along with the Derive keystrokes necessary to explore these problems. We'll also include some pictures of what we are seeing on the computer screen. Ideally, you will type in these examples yourself and then go on to try your own problems. Good luck !

Everyone seems to believe that 2x+3x = 5x even without instruction in algebra. Almost everyone, with this same experience, seems to believe that 2x*3x = 6x. As math teachers how do we cope with this conflict between our students' logical instincts and the rules of algebra? Before Derive, we would explain that x*x or x*x*x or x*x*x*x can be written in a shorthand way . . . x^2 or x^3 or x^4. So 2x*3x might be written 2*3*x*x which could be $6x^2$. Now that we have Derive we still do the same kind of teaching but we use Derive to provide the answer to 2x*3x, which "seems to be" $6x^2$. We suggest

that our students also try 2y*3y or 2a*3a to see if there is a consistent pattern to Derive's answers. We also suggest, when our students have a bit more experience, that they try 2a*3b*4c and 2a*3a*4a and 2*(3a+4b). We also suggest, very early, that all of our students make up their own problems, and that they notice how Derive transforms their expressions. We encourage our students to show us and others what has happened in their work and where they are surprised by the results. We also suggest more complex expressions for experimentation. This may involve solving equations, or factoring algebraic expressions (or numbers), or plotting graphs. Using the computer program to show the inherent logic of the math by producing consistent results is one of our major applications for Derive.

Another important use of Derive is simply as a source of answers. Like some amazingly powerful calculator, Derive will produce answers to numerical and algebraic problems. What are the roots of $x^3-1=0$ or what is the graph of $2\sin(3x)-1=y$ or is 123454321 a prime number and if not what are the factors? Like a trusted 19th century servant or a medieval knight or a genie from a special bottle, Derive is ready to tackle many mathematical jobs for its owner.

Here is a series of sample problems that a user with Derive can easily solve, along with the keystrokes that carry out these solutions. This is a quick way to get an idea of Derive's possibilities. This list can be a good introduction for a user.

Sample Problems and Solutions and Keystrokes to the Solutions

1. Factor $x^2-12x+35$ into linear factors in x.

keystrokes: A (for Author), type x^2–12x+35, Enter, F (for Factor), Enter, R (for Rational).

```
        2
1:     x  - 12 x + 35
2:    (x - 7) (x - 5)
```

2. Factor 123454321 into prime factors.

keystrokes: A (for Author), type 123454321, Enter, F (for Factor), Enter.

```
1:    123454321
        2    2
2:    41  271
```

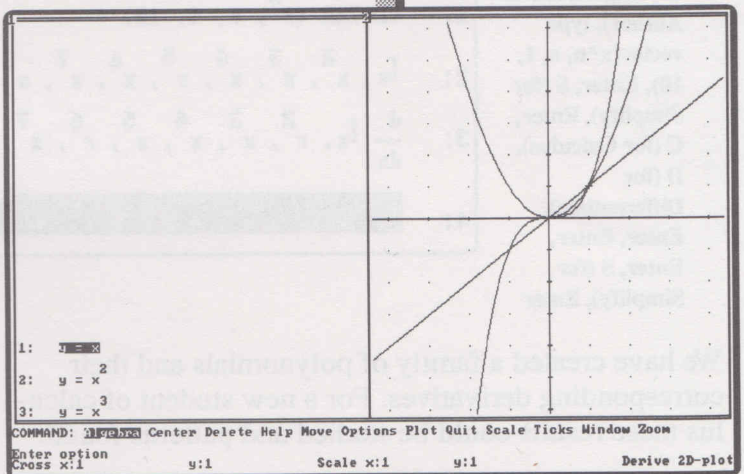
3. Solve $x^2-23x+132 = 0$ for all roots.

keystrokes: A (for Author), type x^2–23x+132=0, Enter, L (for soLve), Enter.

```
       2
1:    x  - 23 x + 132 = 0
2:    x = 11
3:    x = 12
```

4. Graph $y = x$, $y = x^2$, $y = x^3$ on the same axes.

keystrokes: A (for Author), type y=x, Enter, A (for Author), type y=x^2, Enter, A(for Author), type y=x^3, Enter, W (for Window), S (for Split), V (for Vertical), Enter, type key F1 (to move to the graph window), W (for Window), D (for Designate), type 2 (for 2D-plot), y (for yes), P (for Plot), A (for Algebra), arrowup, P (for Plot), P (for Plot), A (for Author), arrowup, P (for Plot),P (for Plot).



If your 2D graphs ever look raggedy, correct this by typing O (for Options), D (for Display), G (for Graphics), Enter. To close a window type W (for Window), C (for Close), Enter.

5. Multiply out (y-13)(y+5)

keystrokes: A (for Algebra), A (for Author), type (y–13)(y+5), Enter, E (for Expand), Enter.

1: $(y - 13)(y + 5)$

2: $y^2 - 8y - 65$

6. Solve $x^6 - 1 = 0$

keystrokes: A (for Author), type x^6–1=0, Enter, L (for soLve), Enter.

1: $x^3 - 1 = 0$

2: $x = 1$

3: $x = -\frac{1}{2} - \frac{\sqrt{3}\,\hat{\imath}}{2}$

4: $x = -\frac{1}{2} + \frac{\sqrt{3}\,\hat{\imath}}{2}$

7. Use Derive to make lists of mathematical objects that can help our mathematical understanding.

keystrokes: A (for Author), type vector(kn, k, 1, 10), Enter, S (for Simplify), Enter.

1: VECTOR (k n, k, 1, 10)

2: [n, 2 n, 3 n, 4 n, 5 n, 6 n, 7 n, 8 n, 9 n, 10 n]

keystrokes: A (for Author), type vector(x^n, n, 1, 10), Enter, S (for Simplify), Enter, C (for Calculus), D (for Differentiate), Enter, Enter, Enter, S (for Simplify), Enter

1: VECTOR $(x^n, n, 1, 10)$

2: $[x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}]$

3: $\frac{d}{dx}[x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}]$

4: $[1, 2x, 3x^2, 4x^3, 5x^4, 6x^5, 7x^6, 8x^7, 9x^8, 10x^9]$

We have created a family of polynomials and their corresponding derivatives. For a new student of calculus these results could be studied and patterns found.

Another kind of list can be created by the command iterates. If we take x = 0.3, and calculate the cos(0.3) and cos(cos(0.3)) and so on . . . we are doing by hand what iterates does automatically.

keystrokes: A (for Author), type iterates(cosx, x, 0.3, 4), Enter, X (for approX), Enter.

```
1:   ITERATES (COS (x), x, 0.3, 4)
2:   [0.3, 0.955336, 0.577334, 0.837920, 0.669009]
```

Notice that we generated five terms when the last option was four. Try it yourself with the last option set for eight. Can you predict a pattern?

If we want to study powers of a number, for example 2:

keystrokes: A (for Author), type vector(2^n, n, 12), Enter, S (for Simplify), Enter.

$$
\begin{array}{ll}
1: & \text{VECTOR}\ (2^n,\ n,\ 12) \\
2: & [2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ 256,\ 512,\ 1024,\ 2048,\ 4096] \\
3: & \text{VECTOR}\ ([\text{SIN}\ (n\ 30\ °),\ \text{COS}\ (n\ 30\ °),\ \text{TAN}\ (n\ 30\ °),\ \text{COT}\ (n\ 30\ °)],\ n,\ 0,\ 6)
\end{array}
$$

We can look at this list and find patterns . . . add up the digits in each number.

We can generate a list of trigonometric values in a similar fashion:

keystrokes: A (for Author), type vector([sin(n*30deg), cos(n*30deg), tan(n*30deg), cot(n*30deg)], n, 0, 6), Enter, S (for Simplify), Enter.

$$
4:\ \begin{bmatrix}
0 & 1 & 0 & \frac{1}{0} \\
\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{3} & \sqrt{3} \\
\frac{\sqrt{3}}{2} & \frac{1}{2} & \sqrt{3} & \frac{\sqrt{3}}{3} \\
1 & 0 & \frac{1}{0} & 0 \\
\frac{\sqrt{3}}{2} & -\frac{1}{2} & -\sqrt{3} & -\frac{\sqrt{3}}{3} \\
\frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{3} & -\sqrt{3} \\
0 & -1 & 0 & \frac{1}{0}
\end{bmatrix}
$$

If you want to study binomial expansions you could try:

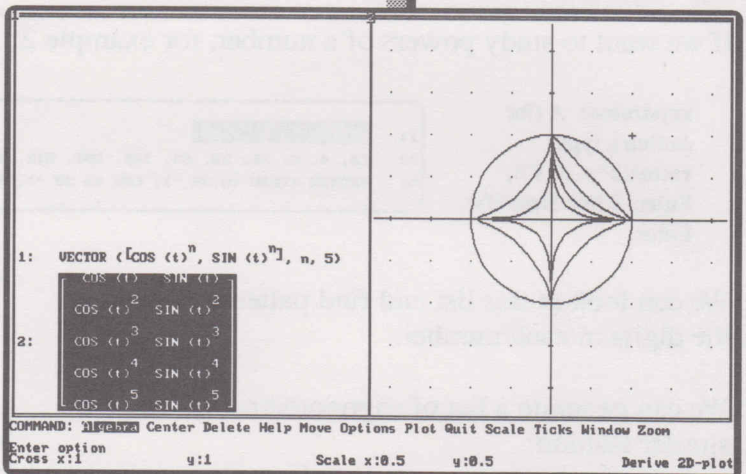
keystrokes: A (for Author), type vector((a+b)^n, n, 5), Enter, S (for Simplify), Enter, E (for Expand), Enter twice.

$$
\begin{array}{ll}
1: & \text{VECTOR}\ ((a + b)^n,\ n,\ 5) \\
2: & [a + b,\ (a + b)^2,\ (a + b)^3,\ (a + b)^4,\ (a + b)^5] \\
3: & [a + b,\ a^2 + 2\ a\ b + b^2,\ a^3 + 3\ a^2\ b + 3\ a\ b^2 + b^3,\ a^4 + 4\ a^3\ b + 6\ a^2\ b^2 + 4\ a\ b^3 +
\end{array}
$$

Since this last list is longer than one line, hold down the Ctrl key and tap arrowright to scroll to the right. Or, tap arrowright, to highlight the first expression, and arrowright again to see the next term and so on.

If you want to generate a list of expressions which can then be graphed, the vector command will do it. We'll create a list of parametric expressions (the first term is the x value and the second term is the y value). The plot command will allow us to plot a list of expressions.

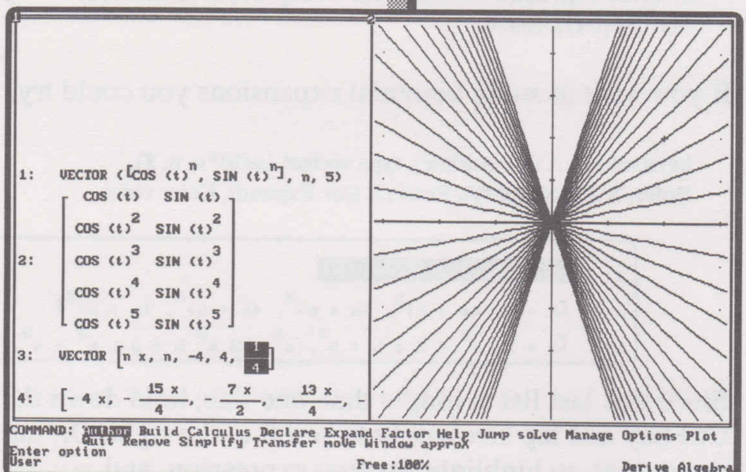
> **keystrokes: A (for Author), type vector([(cost)^n, (sint)^n], n, 5), Enter, S (for Simplify), Enter, W (for Window), S (for Split), V (for Vertical), Enter, tap key F1 (to move to the second window), W (for Window), D (for Designate), 2 (for 2D-plot), Y (for Yes), P (for Plot), tap Enter 5 times, tap key F9 (to zoom in).**



The plot five times is a temporary work-around; it allows us to plot more than one parametric at a time.

You can also generate a list of regular Cartesian plots and plot them all at once. Many nice effects are possible:

> **keystrokes: A (for Algebra), A (for Author), type vector(n*x, n, –4, 4, 1/4), Enter, S (for Simplify), Enter, P(for Plot), D (for Delete), A (for All), P (for Plot).**



Vector is helpful for making a list of decimals, for examination, with enough decimal places to see what's happening.

keystrokes: A (for Algebra), A (for Author), type vector([n, 1/n], n, 23, 33, 1), Enter, O (for Options), P (for Precision), Tab, type 33, Enter, X (for approXimate), Enter, tap key F1, W (for Window), C (for Close), Enter.

$$\text{5:}\quad \text{VECTOR}\left[\left[n, \frac{1}{n}\right], n, 23, 33, 1\right]$$

$$\text{6:}\quad \begin{bmatrix} 23 & 0.0434782608695652173913043478260869 \\ 24 & 0.041666666666666666666666666666666 \\ 25 & 0.04 \\ 26 & 0.0384615384615384615384615384615384 \\ 27 & 0.0370370370370370370370370370370370 \\ 28 & 0.0357142857142857142857142857142857 \\ 29 & 0.0344827586206896551724137931034482 \\ 30 & 0.033333333333333333333333333333333 \\ 31 & 0.0322580645161290322580645161290322 \\ 32 & 0.03125 \\ 33 & 0.0303030303030303030303030303030303 \end{bmatrix}$$

After changing the Precision to a high value, don't forget to change it back to 6 digits, the default case.

A vector of vectors is a matrix, in Derive. Normal matrix operations are available. We may define a matrix, m, two ways:

keystrokes: A (for Author), type m:=[[1,2], [3,4]], Enter, tap D (for Declare), M (for Matrix); if the word Insert is on lower part of the screen tap Insert, type 2; Tab, type 2, Enter, type w, Enter, type x, Enter, type y, Enter, type z, Enter, A (for Author), type p:=, tap key F3 (to copy), Enter, A (for Author), type 2m, Enter, S (for Simplify), Enter.

$$\text{1:}\quad M := \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\text{2:}\quad \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$\text{3:}\quad P := \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

$$\text{4:}\quad 2\ M$$

$$\text{5:}\quad \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

Try lots of combinations . . . m^2 or m+p or m.m (use a decimal point for matrix multiplication), 1/p or p^{-1} for inverses or 2p+3m. In each case S (for Simplify) will carry out the action.

If we are investigating polynomials of the form x^n-1, where n is a natural number, and how these polynomials factor, we might create the following vector and act on it:

keystrokes: A (for Author), type vector(x^n–1, n, 4), Enter, S (for Simplify), Enter, F (for Factor), Enter, R (for Rational)

```
3:   VECTOR (x^n - 1, n, 4)
4:   [x - 1, x^2 - 1, x^3 - 1, x^4 - 1]
5:   [x - 1, (x - 1) (x + 1), (x - 1) (x^2 + x + 1), (x - 1) (x + 1) (x^2 + 1)]
```

If we believe there's a pattern to the results when x^2-1, x^4-1, and x^8-1 are factored, then we can try:

keystrokes: A (for Author), type vector(x^(2^n)–1, n, 3), Enter, S (for Simplify), Enter, F (for Factor), Enter, R (for Rational).

```
6:   VECTOR [x^(2^n) - 1, n, 3]
7:   [x^2 - 1, x^4 - 1, x^8 - 1]
8:   [(x - 1) (x + 1), (x - 1) (x + 1) (x^2 + 1), (x - 1) (x + 1) (x^2 + 1) (x^4 + 1)]
```

8. Use the calculus definition of slope to analyze the slope of y = sin(x).

keystrokes: A (for Author), type (sin(x+h)–sinx)/(x+h–x), Enter, C (for Calculus), L (for Limit), Enter, hold down Del key until clear, type h, Enter, type 0, Tab, A (for Above), Enter, S (for Simplify), Enter.

1: $\dfrac{\text{SIN}(x + h) - \text{SIN}(x)}{x + h - x}$

2: $\lim_{h \to 0} \dfrac{\text{SIN}(x + h) - \text{SIN}(x)}{x + h - x}$

3: COS (x)

9. Solve 3x+5 = 21 using step-by-step methods.

keystrokes: A (for Author), type 3x+5=21, Enter, B (for Build), Enter, tap -, arrowleft, arrowdown, arrowright, Enter, D (for Done), S (for Simplify), Enter.

```
1:   3 x + 5 = 21
2:   (3 x + 5 = 21) - 5
3:   3 x = 16
```

keystrokes: B (for Build), Enter, type /, arrowleft, arrowdown, Enter, D (for Done), S (for Simplify), Enter, X (for approXimate), Enter.

$$4:\quad \frac{3x = 16}{3}$$
$$5:\quad x = \frac{16}{3}$$
$$6:\quad x = 5.33333$$

10. Factor x^4+1 to linear factors using complex numbers if necessary.

keystrokes: A (for Author), type x^4+1, Enter, O (for Options), P (for Precision), E (for Exact), Enter, F (for Factor), Enter, R (for Rational).

$$1:\quad x^4 + 1$$
$$2:\quad x^4 + 1$$

Derive returns the same expression that we put in, so no factoring of x^4+1 with only rational numbers is possible. We'll now try Factor- raDical.

keystrokes: F (for Factor), Enter, D (for raDical).

$$1:\quad x^4 + 1$$
$$2:\quad x^4 + 1$$
$$3:\quad (x^2 + \sqrt{2}\,x + 1)\,(x^2 - \sqrt{2}\,x + 1)$$

We see that it was possible to factor if we used radicals. We'll now highlight the 2 factors separately, and try to factor them allowing complex numbers.

keystrokes: arrowleft, F (for Factor), Enter, C (for Complex), arrowright twice (to highlight x^2–(sqrt2)x+1), F (for Factor), Enter, C (for Complex).

$$4:\quad \left[x - \frac{\sqrt{2}\,(-1 + \hat{\imath})}{2}\right]\left[x + \frac{\sqrt{2}\,(1 + \hat{\imath})}{2}\right](x^2 - \sqrt{2}\,x + 1)$$
$$5:\quad \left[x - \frac{\sqrt{2}\,(-1 + \hat{\imath})}{2}\right]\left[x + \frac{\sqrt{2}\,(1 + \hat{\imath})}{2}\right]\left[\left[x - \frac{\sqrt{2}\,(1 + \hat{\imath})}{2}\right]\left[x + \frac{\sqrt{2}\,(-1 + \hat{\imath})}{2}\right]\right]$$

There are so many paths opened up by this single tiny project that I am left breathless!!

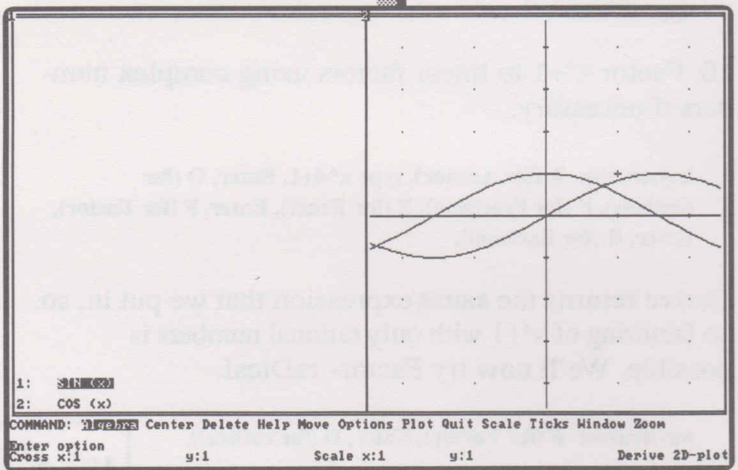
11. Change $(1+\hat{\imath})^{13}$ to normal complex form.

keystrokes: A (for Author), type (1+, hold down the Alt key and type i, type)^13, Enter, S (for Simplify), Enter.

$$1:\quad (1 + \hat{\imath})^{13}$$
$$2:\quad -64 - 64\,\hat{\imath}$$

12. Solve sin(x) = cos(x) by graphical methods.

keystrokes: A (for Author), type sinx, Enter, A (for Author), type cosx, Enter, W (for Window), S (for Split), V (for Vertical), Enter, F1, W (for Window), D (for Designate), type 2 (for 2D-plot), y (for Yes), A (forAlgebra), P (for Plot), P (for Plot), A (for Algebra), arrowup (to highlight sin(x)), P (for Plot), P (for Plot).

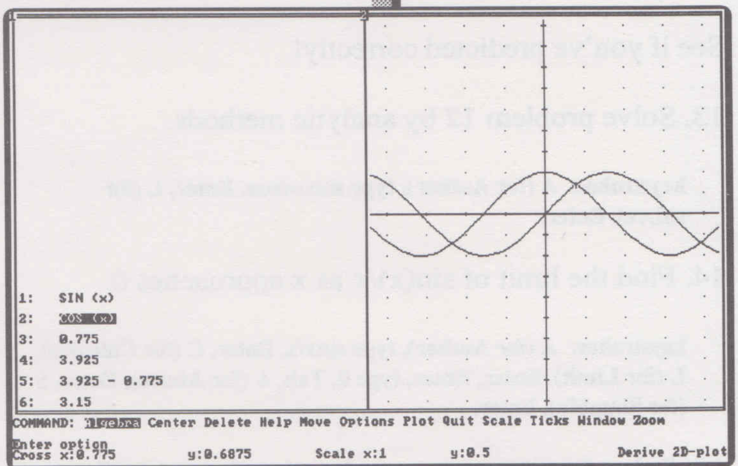


Now you should have 2 graphs on the same grid. The way your picture looks depends on 3 settings: Scale, Option-Accuracy and Ticks. I'll suggest settings now that work nicely and I'll continue through to the end. You might try later to vary the settings, in this context, to see what happens.

keystrokes: S (for Scale), type 1, Tab, type 0.5, Enter, T (for Ticks), tap Del to clear, type 2, Tab, tap Del to clear, type 5, Enter, tap O (for Options), A (for Accuracy), type 6, Enter.

Now we'll move the marker (+) on the screen to a point where the 2 graphs cross. Look on your graph screen and press the arrow keys and you'll probably see the marker moving. If you don't see it, tap the Home key to bring the marker back to (0, 0) and then tap the arrow keys. For faster motion use PgUp and PgDn for the vertical motion and hold down the Ctrl key and tap the arrowleft and arrowright keys for the horizontal.

Arrow to a point where the 2 graphs cross; read the lower left of your screen to learn the x and y values of the marker (+). My reading is x: .775 y: .6785.



keystrokes: A (for Algebra), A (for Author), type .775, Enter, tap key F1 (to return to the graphing window), hold down the Ctrl key and tap arrowright until the marker (+) moves close to the next crossing and then arrowleft or right alone to move closer (if your next crossing point is off the screen, move the marker (+) as close as you can then tap C (for Center)).

My next x reading is 3.925 and the y reading is −.6785.

keystrokes: A (for Algebra), A (for Author), type 3.925, Enter, B (for Build), Enter, type −, arrowup (to highlight 0.775), Enter, D (for Done), X (for approXimate), Enter.

A new game to go with all of the above little bits:

keystrokes: Tap F1 key (to move to the graph window), hold down the Ctrl key and tap arrowleft twice.

This action seems to move the marker (+) to the left 1 unit. If the next crossing is about 2π or 6.28 units to the left of our crossing at (3.925, −.6785) then:

keystrokes: Hold Ctrl and tap arrowleft to move 5 more units to the left (or 10 key presses) and then a few taps on arrowleft alone to come up close to the next crossing.

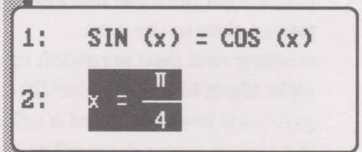
If you're off the screen there is no problem because you then . . .

keystrokes: Type C (for Center)

See if you've predicted correctly!

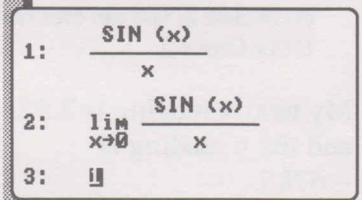
13. Solve problem 12 by analytic methods.

keystrokes: A (for Author), type sinx=cosx, Enter, L (for soLve), Enter.



14. Find the limit of sin(x)/x as x approaches 0.

keystrokes: A (for Author), type sinx/x, Enter, C (for Calculus), L (for Limit), Enter, Enter, type 0, Tab, A (for Above), Enter, S (for Simplify), Enter.

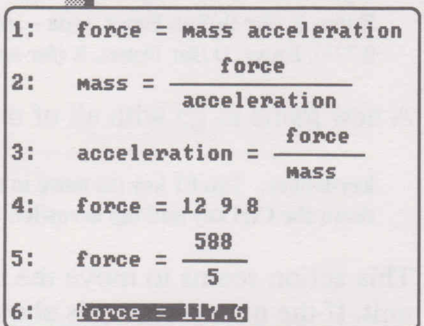


15. Solve force = mass*acceleration for mass; for acceleration; for force if mass = 12 g and acceleration = 9.8 m/sec^2

keystrokes: O (for Options), I (for Input), W (for Word), Enter, A (for Author), type force=mass*acceleration, Enter, O (for Options), P (for Precision), E (for Exact), Enter, L (for soLve), Enter, hold down Del key to clear, type mass, Enter.

keystrokes: L (for soLve), Enter, Enter.

keystrokes: Arrowup (to highlight force=mass acceleration), M (for Manage), S (for Substitute), Enter, hold down Del key to clear and type 9.8 if you just erased acceleration or 12 if you just erased mass, Enter, if SUBSTITUTE variable: is force press Enter, if mass hold Del to clear and type 12, Enter, S (for Simplify), Enter, X (for approXimate), Enter, when you're finished change back to Character Mode . . ., O (for Options), I (for Input), C (for Character), Enter.



16. Factor $64x^6–125y^3$

keystrokes: A (for Author), type 64x^6–125y^3, Enter, F (for Factor), Enter, Enter, R (for Rational).

$$
\begin{array}{ll}
1: & 64\,x^6 - 125\,y^3 \\
2: & (4\,x^2 - 5\,y)\,(16\,x^4 + 20\,x^2\,y + 25\,y^2)
\end{array}
$$

17. Define a function that will give the distance between 2 points, (a,b) and (c,d). Use this function to find the distance between (17, 11) and (–5, –13).

keystrokes: D (for Declare), F (for Function), type distance, Enter, hold down the Alt key and tap Q (for the square root sign), type ((c–a)^2+(d–b)^2), Enter, A (for Author), type distance(17,11,–5,–13), Enter, S (for Simplify), Enter, X (for approXimate), Enter.

$$
\begin{array}{ll}
1: & \mathrm{DISTANCE}\,(a,\ b,\ c,\ d) := \sqrt{(c - a)^2 + (d - b)^2} \\
2: & \mathrm{DISTANCE}\,(17,\ 11,\ -5,\ -13) \\
3: & 2\,\sqrt{265} \\
4: & 32.5576
\end{array}
$$

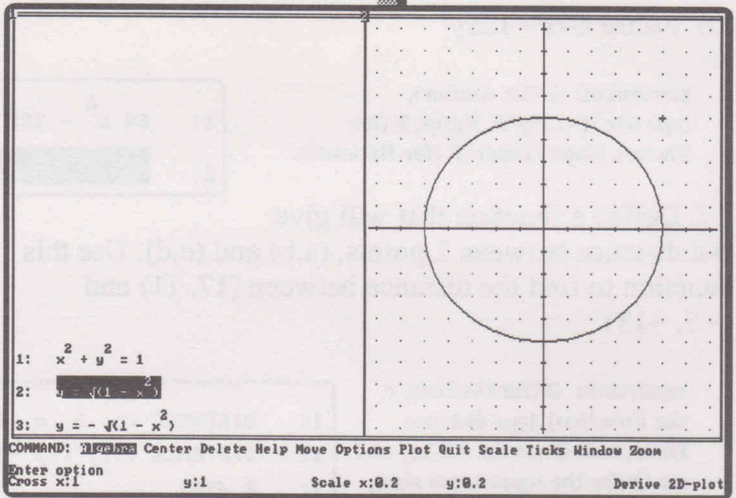
18. Find the slope of 2x–3y = 13 by non–calculus methods; use the form y = mx+b, with m as the slope and b as the y intercept.

keystrokes: A (for Author), type 2x–3y=13, Enter, O (for Option), P (for Precision), E (for Exact), Enter, L (for soLve), Enter, tap Del key until clear, type y, Enter, E (for Expand), Enter, Enter.

$$
\begin{array}{ll}
1: & 2\,x - 3\,y = 13 \\
2: & y = \dfrac{2\,x - 13}{3} \\
3: & y = \dfrac{2\,x}{3} - \dfrac{13}{3}
\end{array}
$$

So, it looks like the slope is 2/3.

19. Graph the circle
$x^2+y^2 = 1$.



keystrokes: A (for Author), type x^2+y^2=1, Enter, L (for soLve), Enter, type y, Enter, W (for Window), S (for Split), V (for Vertical), Enter, tap key F1 (to move to the next window), W (for Window), D (for Designate), 2 (for 2D-plot), y (for yes), P (for Plot).

Now watch Derive graph the top of the circle.

keystrokes: A (for Algebra), arrowup (to highlight y=sqrt(1–x²)), P (for Plot), P (for Plot).

20. Find the sum of the numbers from 1 to 10; from 1 to 20; from 1 to 30; from 1 to n.

1: k

2: $\sum_{k=1}^{10} k$

3: 55

keystrokes: A (for Author), type k, Enter, C (for Calculus), S (for Sum), Enter, Enter, Tab, type 10, Enter, S (for Simplify), Enter.

4: $\sum_{k=1}^{20} k$

5: 210

6: $\sum_{k=1}^{30} k$

7: 465

keystrokes: Arrowup once, A (for Author), tap key F3, hold down Ctrl and tap A twice, type 2, Enter, S (for Simplify), Enter, arrowup once, A (for Author), tap key F3, hold down Ctrl and tap A twice, type 3, Enter, S (for Simplify), Enter.

1: k

2: $\sum_{k=1}^{n} k$

3: $\frac{n\,(n + 1)}{2}$

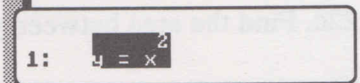
keystrokes: A (for Author), type k, Enter, C (for Calculus), S (for Sum), Enter, Enter, Enter, S (for Simplify), Enter.

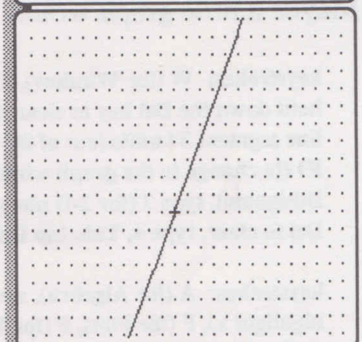
21. Find the slope of $y = x^2$, by graphical methods, at the points (1, 1), (2, 4) and (3, 9). (Here we use Derive as a mathematical microscope!) First, we'll open up a graph window:

> **keystrokes: W (for Window), S (for Split), V (for Vertical), Enter, tap key F1 (to move to the next window), W (for Window), D (for Designate), 2 (for 2D-plot), type y (for yes), A (for Algebra).**

Now we'll work on the graphs.

> **keystrokes: A (for Author), type y=x^2, Enter, P (for Plot), M (for Move), tap down Del to clear, type 1, Tab, tap Del to clear, type 1, Enter, C (for Center), S (for Scale), type 0.001, Tab, type 0.001, Enter, T (for Ticks), type 1, Tab, type 1, Enter, P (for Plot).**

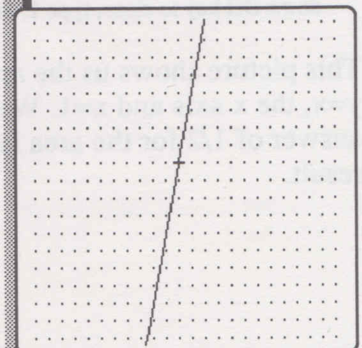




By moving close to the graph $y = x^2$ at (1, 1), we notice that the curve seems rather straight. The marker (+) is at (1, 1) so, by eye, move to the right 1 space (or one grid dot) and up 2 spaces to arrive again at the "curve". The slope is 2/1.

Now we'll move to the point (2, 4) which is also on the curve $y = x^2$.

> **keystrokes: M (for Move), type 2, Tab, type 4, Enter, C (for Center).**



We are now looking closely at the same $y = x^2$ with (2, 4) as our center of focus. Count over to the right 1 and up 4 to meet the "curve" so the slope is 4/1 (the "curve" does seem steeper here at (2, 4) than before at (1, 1)). Try the same process to focus on (3, 9) or anywhere else on the "curve". Try the same process for different functions; for example $y = x^3$ or $y = 2x^2$ or $y = e^x$ or $y = \sin(x)$ or . . .

22a. Find the area between y = x and y = 0 and x = 1.

keystrokes: A (for Author), type x, Enter, C (for Calculus), I (for Integrate), Enter, Enter, type 0, tap Tab, type 1, Enter, S (for Simplify), Enter.

1: x

2: $\int_0^1 x \, dx$

3: $\frac{1}{2}$

Questions 22b and 22c are given without keystrokes. You need to do almost exactly the same as the keystrokes for 22a.

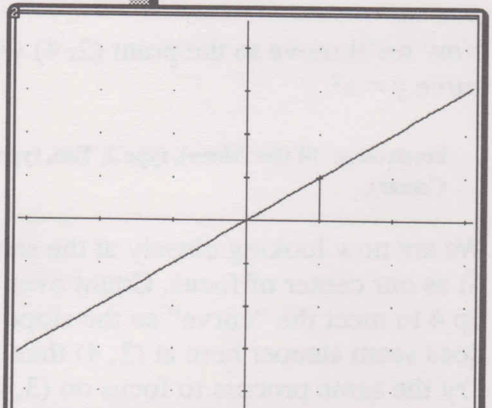
22b. Find the area between $y = x^2$ and y = 0 and x = 1.

22c. Find the area between $y = x^3$ and y = 0 and x = 1.

Maybe a little graph would be nice.

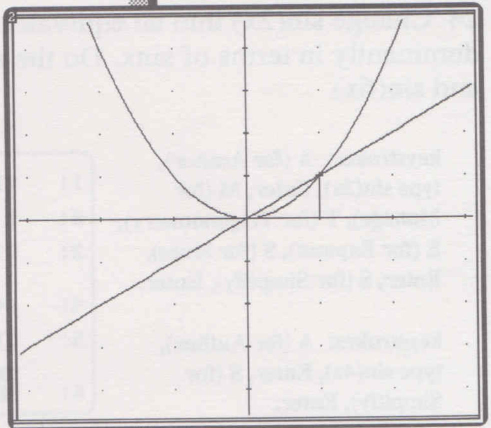
keystrokes: W (for Window), S (for Split), V (for Vertical), hold down the Del key to clear, type 34 (to place the vertical line segment 34 units (out of 80) from the left), Enter, tap key F1 (to change to the graph window), W (for Window), D (for Designate), type 2 (for 2-D plot), y (for yes), T (for Ticks), tap Del to clear, type 4, Tab, tap Del to clear, type 9, Enter.

keystrokes: A (for Algebra), arrowup, arrowright (to highlight x), P (for Plot), P (for Plot), A (for Algebra), A (for Author), type [1,y], Enter, P (for Plot), P (for Plot), hold down Del key to clear, type 0, Tab, hold down Del key to clear, type 1, Enter.



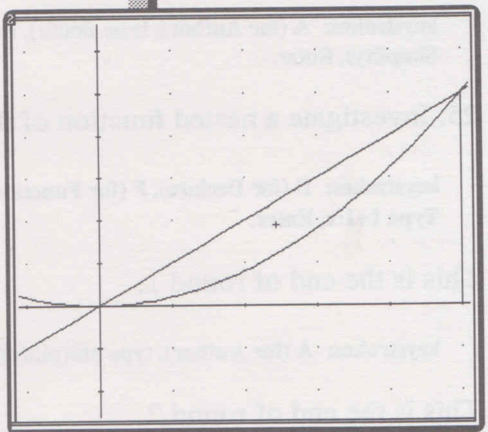
This picture shows us the area enclosed by y=x, the x axis and x=1. We can see that the answer of 1/2 for the area is a reasonable result.

keystrokes: A (for Algebra), A (for Author), type x^2, Enter, C (for Calculus), I (for Integrate), Enter, Enter, type 0, Tab, type 1, Enter, S (for Simplify), Enter, arrowup, arrowright (to highlight x^2), P (for Plot), P (for Plot).



My picture is small so I'll move the marker (+) with the arrow keys to the center of action.

keystrokes: C (for Center), tap key F9, tap key F9, C (for Center) to have a better look.



Try it yourself.

keystrokes: A (for Algebra), A (for Author), type x^3, Enter, C (for Calculus), I (for Integrate), Enter, Enter, type 0, Tab, type 1, Enter, S (for Simplify), Enter, arrowup, arrowright (to highlight x^3), P (for Plot), P (for Plot).

23. Derive can work in base 2 through 36. Here we will change base 10 input to base 2 output and try it on the number 32 and also on factoring $x^4–1$.

keystrokes: O (for Options), R (for Radix), Tab, tap Del to clear, type 2 (for Output), Enter, A (for Author), type 32, Enter, A (for Author), type x^4–1, Enter, F (for Factor), Enter, R (for Rational).

```
1:    100000
       100
2:    x    - 1
                         10
3:    (x - 1) (x + 1) (x    + 1)
```

When you're finished with this exercise change the radix back to 10 for the input and 10 for the output.

24. Change sin(2x) into an equivalent statement predominantly in terms of sinx. Do the same for sin(4x) and sin(6x).

keystrokes: A (for Author), type sin(2x), Enter, M (for Manage), T (for Trigonometry), E (for Expand), S (for Sines), Enter, S (for Simplify), Enter.

keystrokes: A (for Author), type sin(4x), Enter, S (for Simplify), Enter.

```
1:   SIN (2 x)
2:   2 SIN (x) COS (x)
3:   SIN (4 x)
                        3
4:   (4 SIN (x) - 8 SIN (x) ) COS (x)
5:   SIN (6 x)
                 5             3
6:   (32 SIN (x)  - 32 SIN (x)  + 6 SIN (x)) COS (x)
```

keystrokes: A (for Author), type sin(6x), Enter, S (for Simplify), Enter.

25. Investigate a nested function of the form 1+1/x.

keystrokes: D (for Declare), F (for Function), type phi, Enter, Type 1+1/x, Enter.

```
                      1
1:   PHI (x) := 1 + ———
                      x
```

This is the end of round 1.

keystrokes: A (for Author), type phi(phi(x)), Enter.

```
2:   PHI (PHI (x))
```

This is the end of round 2.

keystrokes: M (for Manage), S (for Substitute), Enter, Del to clear, F3 (to copy), Enter.

```
3:   PHI (PHI (PHI (PHI (x))))
```

This is the end of round 3. At the end of round 1 we had 1 level; at the end of round 2 we had 2 levels; at the end of round 3 we had 4 levels and at the end of round 4 we would have 8 levels.

To see a number which is an approximation for the value of phi:

keystrokes: M (for Manage), S (for Substitute), Enter, Del to clear, type 1, Enter, S (for Simplify), Enter, X (for approX), Enter.

```
4:   PHI (PHI (PHI (PHI (1))))
```

$$5:\quad \frac{8}{5}$$

$$6:\quad 1.6$$

This can all be automated:

keystrokes: A (for Author), type iterates(1+1/x, x, 1+1/ x, 5), Enter, F (for Factor), Enter, R (for Rational), M (for Manage), S (for Substitute), Enter, type 1, Enter, S (for Simplify), Enter.

$$1:\quad \mathrm{ITERATES}\left[1 + \frac{1}{x}, x, 1 + \frac{1}{x}, 5\right]$$

$$2:\quad \left[\frac{x + 1}{x}, \frac{2\,x + 1}{x + 1}, \frac{3\,x + 2}{2\,x + 1}, \frac{5\,x + 3}{3\,x + 2}, \frac{8\,x + 5}{5\,x + 3}, \frac{13\,x + 8}{8\,x + 5}\right]$$

$$3:\quad \left[\frac{1 + 1}{1}, \frac{2\,1 + 1}{1 + 1}, \frac{3\,1 + 2}{2\,1 + 1}, \frac{5\,1 + 3}{3\,1 + 2}, \frac{8\,1 + 5}{5\,1 + 3}, \frac{13\,1 + 8}{8\,1 + 5}\right]$$

$$4:\quad \left[2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}\right]$$

26. Investigate $y = x^{10}$, $y = 2^x$ and $y = 3^x$ to see where they might intersect.

keystrokes: A (for Author), type vector([x, x^10, 2^x, 3^x], x, 10, 60, 10), Enter, X (for approX), Enter.

$$1:\quad \mathrm{VECTOR}\left(\left[x, x^{10}, 2^{x}, 3^{x}\right], x, 10, 60, 10\right)$$

$$2:\quad \begin{bmatrix} 10 & 10^{10} & 1024 & 59049 \\ 20 & 1.024\ 10^{13} & 1.04857\ 10^{6} & 3.48678\ 10^{9} \\ 30 & 5.9049\ 10^{14} & 1.07374\ 10^{9} & 2.05891\ 10^{14} \\ 40 & 1.04857\ 10^{16} & 1.09951\ 10^{12} & 1.21576\ 10^{19} \\ 50 & 9.76562\ 10^{16} & 1.12589\ 10^{15} & 7.17896\ 10^{23} \\ 60 & 6.04661\ 10^{17} & 1.15292\ 10^{18} & 4.23881\ 10^{28} \end{bmatrix}$$

From the resulting table we can see the general region for each of the functions crossing. For x^{10} and 2^x we might run the same vector expression between 55 and 60.

I first heard of this problem at the Exeter Conference on Secondary Math Teaching and Computers. I would recommend this stimulating group to any math teacher.

keystrokes: A (for Author), type vector([x, x^10, 2^x], x, 55, 60), Enter, X (for approXimate), Enter.

3: $\text{VECTOR}\left([x, x^{10}, 2^x], x, 55, 60\right)$

4: $\begin{bmatrix} 55 & 2.53295 \cdot 10^{17} & 3.60287 \cdot 10^{16} \\ 56 & 3.03305 \cdot 10^{17} & 7.20575 \cdot 10^{16} \\ 57 & 3.62033 \cdot 10^{17} & 1.44115 \cdot 10^{17} \\ 58 & 4.30804 \cdot 10^{17} & 2.88228 \cdot 10^{17} \\ 59 & 5.11116 \cdot 10^{17} & 5.76460 \cdot 10^{17} \\ 60 & 6.04661 \cdot 10^{17} & 1.15292 \cdot 10^{18} \end{bmatrix}$

From this table it looks like $y=x^{10}$ and $y=2^x$ must cross between x = 58 and x = 59. The chase could be narrowed as far as we like.

27. Integrate Ln(x^n) with n going from 1 to 5 .

Derive is quite good at indefinite integration, as you'll see here. It's also true that many functions "distribute" across vectors (like Factor, Differentiate, Integrate, Sum, Limit, Expand, approXimate and Plot.

keystrokes: A (for Author), type vector(ln(x^n),n,5), Enter, S (for Simplify), Enter, C (for Calculus), I (for Integrate), Enter, Enter, Enter, S (for Simplify), Enter.

1: $\text{VECTOR}\left(\text{LN}\left(x^n\right), n, 5\right)$

2: $\left[\text{LN}(x), \text{LN}\left(x^2\right), \text{LN}\left(x^3\right), \text{LN}\left(x^4\right), \text{LN}\left(x^5\right)\right]$

3: $\int \left[\text{LN}(x), \text{LN}\left(x^2\right), \text{LN}\left(x^3\right), \text{LN}\left(x^4\right), \text{LN}\left(x^5\right)\right] dx$

4: $\left[x\ \text{LN}(x) - x, x\ \text{LN}\left(x^2\right) - 2x, x\ \text{LN}\left(x^3\right) - 3x, x\ \text{LN}\left(x^4\right) - 4x, x\ \text{LN}\left(x^5\right) - 5x\right]$

This series of problems was designed to show the power and versatility of Derive. How you use Derive is important to the author. Whether you are a 5th grader, an engineer or physicist or math teacher, a college or high school or junior high school student, I am interested in your favorite uses for Derive. Please let me know about your progress.

Chapter 2

Calculating

With a computer program like Derive you can have some fun with calculating and may have **insights**. Let's start with an example: 25*16 = 400.

keystrokes: A (for Author), type 25*16, Enter, S (for Simplify), Enter.

```
1:   25 16
2:   400
```

In Derive, multiplication appears on the screen as a space between the two elements. When you are typing a multiplication problem into Derive, you can use either the space or the *.

How do we do this calculation in our head? Well . . . another name for 25 is (1/4)*100 so . . . 25*16 = (1/4)*100*16 = 100*(1/4)*16. So now 25*16 can be done mentally. (1/4)*16 = 4 and 100*4 = 400, so 25*16 = 400.

Let's try the same technique on 25*12 = (1/4)*100*12 = 100*(1/4)*12 = 100*3 = 300.

Try 25*32   25*24   25*44. Make up your own and teach a friend how to do this. Teaching is a great way to learn!

Use Derive to check your mental calculations and also to record your results.

Now that you can do 25*12 in your head, how about 75*12? If 75 = 3*25 then 75*12 = 3*25*12 so 3*((1/4)*100)*12 = 3*((1/4)*12)*100 = 3*3*100 = 3*300 = 900.

Here's 75*28 the same way:

75*28 = 25*3*28 = 3*(1/4)*100*28 = 3*100*7 = 2100.

Try 75*24 75*36 75*44.

Make up your own. Use Derive to check and record your results.

If 25*12 = 300, then 25*13 must be 25*(12+1) = 25*12+25*1 = 300+25 = 325.

Try 25*17 25*25 25*33 25*41 25*101 25*89.

What about 25*15?

If 25*16 = 400, then 25*15 = 25(16–1) = 400–25 = 375.

Try 25*23 25*31 25*43 25*7 25*47 25*87.

And what about 25*18?

25*16 = 400, so 25*18 = 25(16+2) = 400+50 = 450.

Try 25*26 25*34 25*46 25*54 25*22 25*66.

Could we do 75*23 mentally?

75*24 = 3*25*24 = 3*600 = 1800. So . . . 75*23 = 75(24–1) = 1800–75 = 1725.

Try . . . 75*17 75*25 75*31 75*37.

Now we'll mix them all together . . . see if you can sort them out. 25*48 75*36 25*13 75*14 75*48 25*49 75*49.

Practice these for five minutes a day and they will become simple to do. Teach a friend how to do it and

YOU will improve. Look for patterns like these and write to us and share your discoveries.

New patterns of calculation . . . 9*11   19*21   29*31. Use Derive to record these statements, do your mental calculations, and use Derive's answers to check you work.

keystrokes: A (for Author), type 9*11, Enter, S (for Simplify), Enter and so on for the others.

```
3:   9 11
4:   99
```

Let's pause and look at the pattern of the statements: 9*11   19*21   29*31 and so on. What would be next? And after that? And then?

Now look at the pattern in the products.

Guess the next result! 39*41 = ????

Well . . . 99 399 899 seems to indicate the calculations all will end in 99. Our next job is to look for a pattern in the digit(s) in front of the 99. 0 3   8. Since 0 to 3 is 3, and 3 to 8 is 5, maybe the next step is 7. 99, 399, 899, (8+7) and then 99 producing the next number, 1599. So 39*41 = 1599. Next 15+9 = 24, so 49*51 = 2499 . . . Keep going.

Try 8*12   18*22   28*32. Use Derive to build up this record and then guess the next and record your guess.

keystrokes: A (for Author), type 38*42=, type your guess here, Enter, S (for Simplify), Enter.

```
5:   38 42 = 1596
6:   1596 = 1596
```

Keep going with the next question in the sequence and your guess. As always, there's more than one pattern here. The more you look, the more you'll find.

Try 7*13   17*23   27*33 and so on.

Try $(10-1)(10+1) = 10^2-1^2$ $(20-1)(20+1) = 20^2-1^2$.

$(n-1)(n+1) = n^2-1^2$. $(n-a)(n+a) = n^2-a^2$.

Patterns and Calculating with Powers.

$3^2+4^2 = 5^2$ seems to be true.

> **keystrokes: A (for Author), type 3^2+4^2 = 5^2, Enter, S (for Simplify), Enter.**

```
7:  3^2 + 4^2 = 5^2
8:  25 = 25
```

Try $4^2+5^2 = 6^2$ $2^2+3^2 = 4^2$ $6^2+8^2 = 10^2$ $29^2+30^2 = 31^2$ $9^2+12^2 = 15^2$ $n^2+(n+1)^2 = (n+2)^2$

Which ones were TRUE? Follow up . . . push it further.

Try $(3n)^2+(4n)^2 = (5n)^2$ $1^3+2^3+3^3 = (1+2+3)^2$
$(1^1+2^3+3^3+4^3) = (1+2+3+4)^2$ $(n-1)^3+n^3+(n+1)^3 = (3n)^2$

Fractions and Decimals.

Derive is a great place to experiment with fractions and decimals. If 1/4 = .25 then 2/4 must be .50 and 3/4 must be .75 Who needs a computer? Who knows 1/8 as a decimal? Not everyone. But everyone knows 2/8 = .25 because 2/8 = 1/4.

So . . . 1/8 is half of 2/8 and therefore 1/8 = 1/2 of .25 or 1/2 of .250 = .125. Now 3/8 is easy . . . it's 1/8 + 2/8 or .125 + .250 = .375. And so on.

Who knows 1/12 as a decimal? Not too many. 2/12? Still a problem. 3/12? No sweat! 3/12 = 1/4 = .25 or

.250 so 1/12 is .250/3 = .0833 or (8 and 1/3) hundredths so 2/12 is .16666 or (16 and 2/3) hundredths and so on.

Try the same process on 1/16, 2/16 . . . (The trick we learned with twelfths is to try an early problem, say 1/12, and if it seems hard, then move on to 2/12, which we also don't know, and then go to 3/12 which we do know. Don't stop at the fraction you don't know . . . move on until you find one you do know and then work backwards (this gives hindsight). Now you're ready to do the 16ths.

Try 1/7 as a decimal. Who knows? 2/7? 3/7?

4/7? . . . this is not getting better. We need a new approach.

keystrokes: A (for Author), type 1/7, Enter, S (for Simplify), Enter, X (for approXimate), Enter.

```
9:   1
     —
     7
10:  0.142857
```

This result of 1/7 to 6 decimals is not exactly what we want. Maybe 80 decimals or so would be better.

keystrokes: O (for Options), P (for Precision), Tab, tap Del to clear, type 80, Enter, arrowup to highlight 1/7, X (for approXimate), Enter.

```
11:  0.14285714285714285714285714285714285714285714285714285714285714285714285714285714
```

Now we can see what's happening. Try 2/7 and 3/7 and look closely at the decimals in each case. Patterns, patterns . . . guess 4/7 and 5/7 and so on. Say the digits in 1/7 out loud . . . listen to yourself . . . one, four, two, eight, five, seven, one, four, two, eight . . . NOTICE . . . NOTICE. You can do it. It's like a secret world that has been there waiting for your eyes to see. PREDICT

. . . PREDICT . . . and check. Try 1/9, 2/9 . . . 1/11, 2/11 . . . 1/17, 2/17 . . . 1/37, 2/37 . . . Keep going!

More Fractions and Decimals With Another Look

Certain fractions produce interesting results.
1/3 = .33333333333 . . .

The three dots at the end mean that it continues in the same way. Derive can help with these patterns.

> **keystrokes: A (for Author), type 1/3, Enter, O (for Option), P (for Precision), Tab, tap Del key to clear, type 12, Enter, X (for approXimate), Enter.**

```
12:  1/3

13:  0.333333333333
```

As you can see, we can control how many digits appear. So 1/3 has 1 digit repeating as a decimal. Look for other fractions whose decimal representation is 1 digit repeating. Can you find fractions whose decimal seems to be just 2 digits repeating? 3 digits? 5 digits repeating?

Let's turn the question around. If I present a decimal like 0.37, can you make that into a fraction? Easy: 37/100. How about .3737? Easy again: . . . 3737/10000. OK, how about 0.3737 . . . (This means 0.37 continues forever)? It's hard to believe there is a simple fraction which equals such a decimal. Go hunt for it. Use Derive to help your search. Look for a long time and you'll learn a lot.

Now that you've had your fun hunting (with or without success) here is the lovely analysis that came about in a conversation at Urbana High School in Urbana, Illinois. I was preparing a class of future high school math teachers at the University of Illinois and we were work-

ing each day at the local high school. Sometimes we worked with high school students and sometimes among ourselves. A university student said he'd forgotten the procedure for changing repeating decimals to a fraction, I said, Let's invent a new way (Maybe I'd forgotten the standard procedure, too.).

We looked at 0.3737 . . . and asked what's an approximate answer? We immediately came up with 37/100. We asked, "Is this too big or too small?" Well 37/100 is 0.37000, so it's smaller than 0.3737 . . .

OK, how do we make fractions bigger? Just two ways: make the top number bigger (in this case, 38/100–which is too big) or make the bottom number smaller (37/99). We were both shocked to find that 37/99 was exactly correct.

keystrokes: A (for Author), type 37/99, Enter,
X (for approXimate), Enter.

14: $\frac{37}{99}$

15: 0.373737373737

Wow! Does that always work?

Try these infinitely repeating decimals. See if you can find fractions that equal them.

Try 0.2424 . . . or 0.1313 . . . or 0.9797 . . . or 0.0707 . . . or 1.6262 . . . or 0.379379 . . . or 0.0264264 . . . or 13.0043924392 . . . They can be done. Were we excited!

That's the real reason for studying math: the excitement that comes when an idea that is new to you appears. You never know when. Let's see: 1.6262 . . . would be 1 + 62/100 = 1.62, so 1 + 62/99 should do it.

Let's try 0.0264264. Start with 264/1000 = 0.264. So 264/999 = 0.264264 . . . If I now divide by 10, then 264/9990 might do it. Make up new and better ways. Write to us when you find one. It's great when you make your own discoveries.

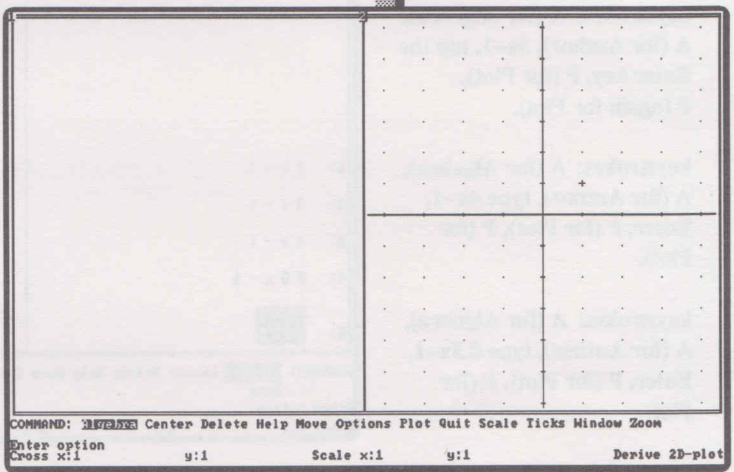
Our precision has been set for 12 digits. A good habit to develop is to change the precision back to 6 digits as soon a possible. Otherwise, you may be graphing a function and Derive will take much longer than necessary because the precision is set too high.

Chapter 3

Graphs, Graphs, Graphs!

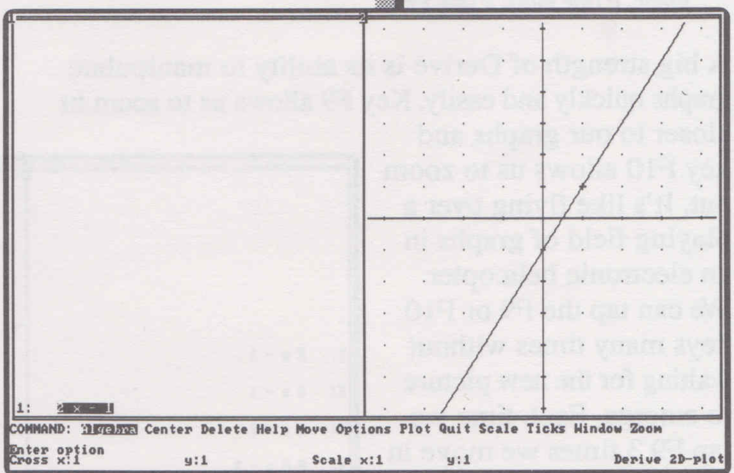
Graphs are an important part of math exploration, and Derive can help. First we'll show you how to set up a most useful screen arrangement.

keystrokes: W (for Window), S (for Split), V (for Vertical), tap the Enter key, tap key F1 (across the top or at the left edge of your keyboard), W (again for Window), D (for Designate), type 2 (for 2D–Plot), O (for Options), D (for Display), G (for Graphics), H (for High), choose your monitor, Enter.



This arrangement of an algebra screen at the left and a graph screen at the right is the setup that we use 90% of the time. Derive is capable of supporting many separate windows; I don't suggest that you do this until you have more experience.

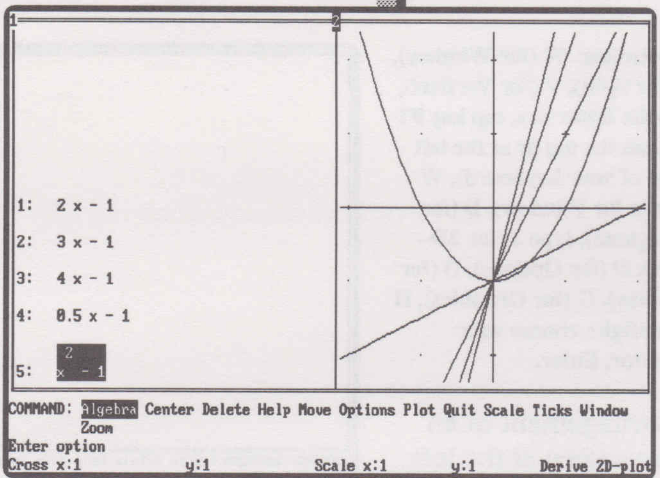
keystrokes: A (for Algebra), A (for Author), type 2x–1, tap the Enter key, P (for Plot), P (again for Plot).



Here you see the graph of y = 2x–1 or f(x) = 2x–1 depending on which notation you prefer. Notice that the graph crosses the vertical or y axis at the point (0,–1) and the pattern on the graph is to the right 1 tick and up 2 ticks. Try graphing

3x–1 and 4x–1 and 0.5x–1 to see what pattern continues and to get comfortable with making graphs with Derive.

keystrokes: A (for Algebra), A (for Author), 3x–1, tap the Enter key, P (for Plot), P (again for Plot).

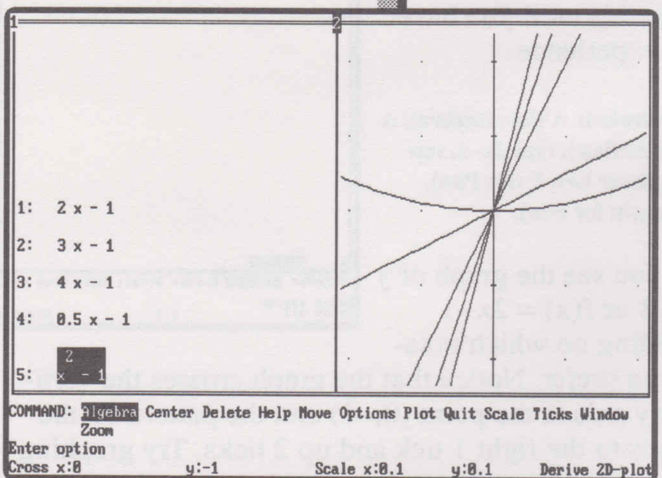


keystrokes: A (for Algebra), A (for Author), type 4x–1, Enter, P (for Plot), P (for Plot).

keystrokes: A (for Algebra), A (for Author), type 0.5x–1, Enter, P (for Plot), P (for Plot).

keystroke: A (for Algebra), A (for Author), type x^2–1, Enter, P (for Plot), P (for Plot).

A big strength of Derive is its ability to manipulate graphs quickly and easily. Key F9 allows us to zoom in closer to our graphs and key F10 allows us to zoom out. It's like flying over a playing field of graphs in an electronic helicopter. We can tap the F9 or F10 keys many times without waiting for the new picture to emerge. Each time we tap F9 3 times we move in by a factor of 10. We can



get the same change in the opposite direction for 3 taps on F10. The scale reading under your graph may be at 1 now and if you tap F9 3 times quickly the reading should become 0.1 which indicates a move in by a factor of 10. Try it now.

> **keystrokes: Tap key F9 (to move closer), tap key F9, now tap key F10 (to back away).**

Try tapping the F9 and F10 keys slowly, allowing the picture to redraw and then try tapping the keys quickly twice or 5 times and see if the result is what you expect.

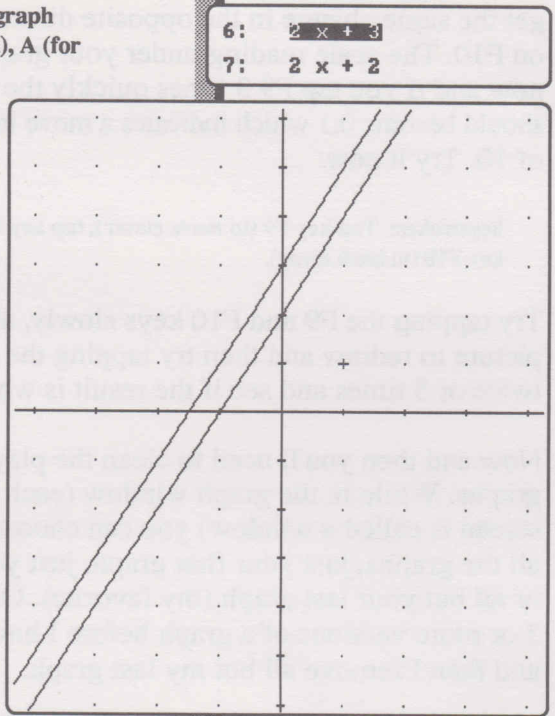
Now and then you'll need to clean the playing field of graphs. While in the graph window (each section of the screen is called a window) you can choose to clear off all the graphs, just your first graph, just your last graph, or all but your last graph (my favorite). Often, I try 2 or 3 or more versions of a graph before I have what I want and then I remove all but my last graph.

To work in a window, Derive must be active in that window; look at the upper left corner of each window. The shaded number indicates the active window. (On most machines, the number for the **active** window appears on a black background. On some machines, however, the number will appear to be highlighted or brighter and will not have a black background.) You can move forward among windows by tapping key F1. You can move backwards among windows by holding down the Crtl key and tapping key F1. Try it now.

Now you'll try new graphs; the first directions will assure a clean field.

keystrokes: Use F1 key until you are active in the graph window, D (for Delete), A (for All), A (for Algebra), A (for Author), type 2x+3, Enter, P (for Plot), P (for Plot).

keystrokes: A (for Algebra), A (for Author), type 2x+2, Enter, P (for Plot), P (for Plot).



If this does not produce graphs that you can see, then it may be producing graphs outside the present scale of the screen. Tap key F10 to back up until the scale number on the screen reaches 1. Then the graphs should come into view.

I have to tell you that many times I stared at the screen wondering what happened to my graphs. A glance at the scale numbers may help. F9 makes the scale numbers smaller and moves us closer and F10 makes the scale numbers bigger and moves us further away.

Did you notice that the graphs were parallel, that 2x+3 went through the point (0, 3), and that 2x+2 went through the point (0, 2)? You can locate any point on the graph by moving the marker (+) around the screen with the arrow keys (when you are in the graph window).

keystrokes: Look at the upper left of the graph window and see if the number 2 is shaded. . . if yes, carry on. . . if not, tap key F1; tap the arrow keys up and down and left and right and watch the marker (+) move on the graph screen. If you like faster action, tap the PgUp and PgDn keys for bigger jumps; hold the Ctrl key down and tap the arrow keys left and right for more of the same.

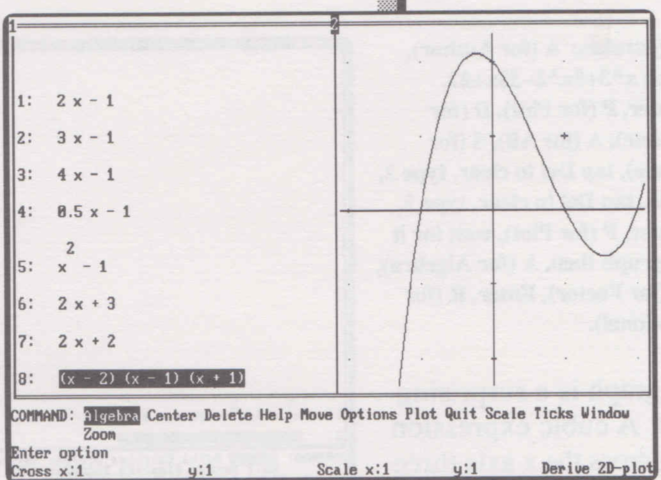
After you've seen how you can move on the screen, look to the lower left part of your screen which has symbols:

Cross x:1   y:1

Your numbers will probably be different from mine. These numbers are the x and y values (coordinates) of the current position of the marker (+) on your screen. This gives us a remarkable tool. For any point on any graph that you can locate by eye (easy to do . . . just look at the graph and move the marker (+) to that point) you can determine easily what the x and y coordinates are at this point. Wow! There must be hundreds of uses for this feature.

Graph (x–2)(x–1)(x+1) = y and locate where this graph crosses the x or horizontal axis.

keystrokes: Use F1 to make the graph window active, D (for Delete), A (for All), A (for Algebra), A (for Author), type (x–2)(x–1)(x+1), tap the Enter key, P (for Plot), P (for Plot).



Once your graph appears, tap F9 or F10 until the scale numbers are 1. Then tap the arrow keys to move the marker (+) to the places where the graph crosses the x axis.

If all works well above, you might see a connection between the x number of the crossing point and the numbers in the expression.

Opportunities:

Make an expression whose graph crosses the x or horizontal axis at 2 and −2. Then check your result by plotting.

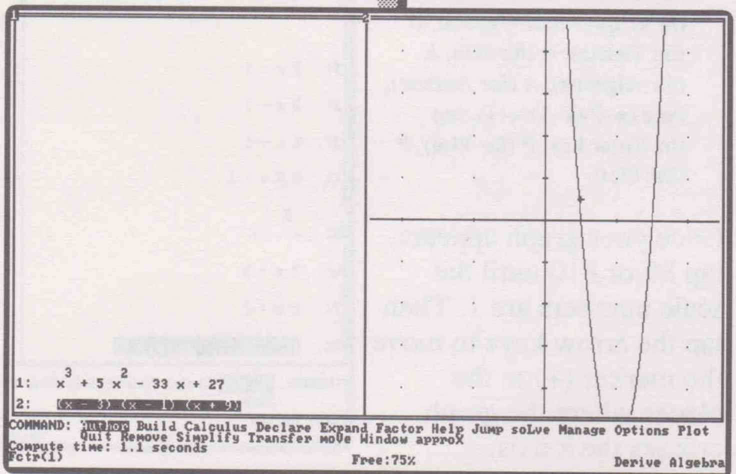
Graph x^2-4x+3. Locate where this graph crosses the x axis.

Graph $-x^2 +3$. Locate the crossing points on the x and y axes.

Do these numbers relate to the equation? Ideas?? Can you think of your own experiments? Derive is waiting to assist you on your own path to learning.

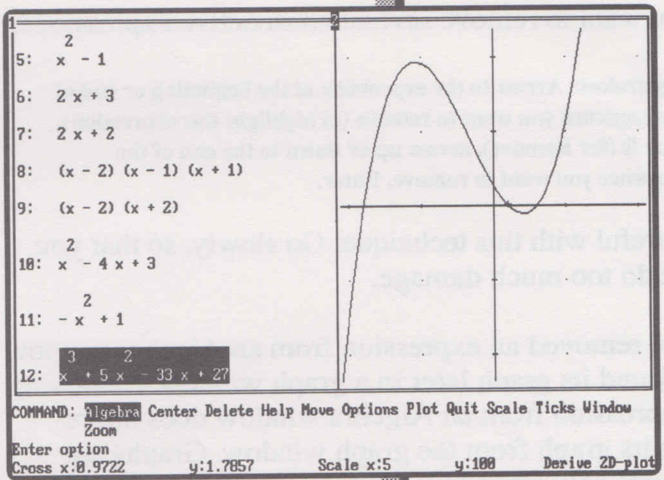
Graph $x^3+5x^2-33x+27$ and see where the graph crosses the x axis.

keystrokes: A (for Author), type x^3+5x^2–33x+27, Enter, P (for Plot), D (for Delete), A (for All), S (for Scale), tap Del to clear, type 2, Tab, tap Del to clear, type 2, Enter, P (for Plot), wait for it to graph then, A (for Algebra), F (for Factor), Enter, R (for Rational).



This graph is a surprising result. A cubic expression might cross the x axis three times or might cross it once and be tangent to it once, or might cross it once . . . but not cross it twice like this graph. Now we'll back away, by changing the scale, to try to clear up this mystery.

keystrokes: Tap key F1 (to move to next window), S (for Scale), tap Del to clear, type 5, Tab, tap Del to clear, type 100, Enter.



By changing the scale we see the real picture of this function.

Try creating an expression whose graph crosses the x axis a few times close together and at another point not so close.

Choose a point like (2,3). Move the marker (+) to this spot. Make up an expression whose graph will go through this designated target point. Try lots of expressions . . . don't be discouraged if you miss a lot . . . keep trying . . . you will get very good at this with practice. Move the target point around. Play this game every day for a month.

Try to get 10 graphs through a single point.

Try to create a graph which goes through 2 chosen points.

Can you create more than one graph that will go through 2 chosen points?

Remove and Delete commands are important for the efficient use of Derive. If you're active in an Algebra window, Remove allows you to remove one or many expressions. If you want to remove a single expression:

keystrokes: Arrow to the expression you want to remove (to highlight it), type R (for Remove), Enter.

If you want to remove several consecutive expressions:

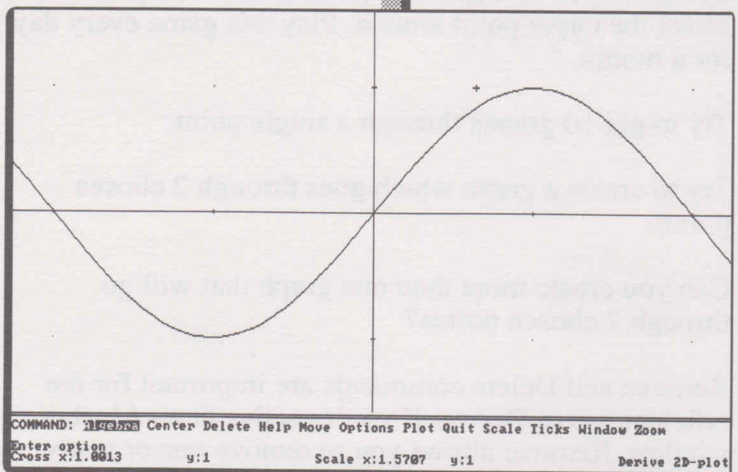
> **keystrokes: Arrow to the expression at the beginning or end of the sequence you want to remove (to highlight this expression), type R (for Remove), arrow up or down to the end of the sequence you want to remove, Enter.**

Be careful with this technique. Go slowly, so that you don't do too much damage.

I have removed an expression from an Algebra window and found its graph later in a graph window. Removing an expression from an Algebra window does not remove its graph from the graph window. Graphs are removed only when you are active in the graph window, and by using the Delete command along with All or Butlast or Last or First.

Derive is prepared to display a great variety of graphs:

> **keystrokes: Tap key F1 (to make the graph window active), W (for Window), C (for Close), Enter, A (for Author), type sinx, Enter, P (for Plot), S (for Scale), tap Del to clear, hold down Alt key and type P, type /2, Enter, P (for Plot).**



Yes, all kinds of graphs are possible . . . sine is typed in as sin, cosine as cos, tangent as tan, cotangent as cot, secant as sec and cosecant as csc. If you want sine of 2x you need sin(2x). Try sec (2.4x−p) or 2cos(3x−3.14)+1. If you want 2x−1 try it just that way.

Try sqrt(9−x^2) and −sqrt(9−x^2) . . . lots and lots of possibilities. Try . . . try . . .

Tips:

- Use F9 or F10 to see if your graph is hidden from you by being too close or too far away.
- Use delete to clear away the debris.
- In the Algebra window, retrieve expressions written earlier using the arrow keys and/or Pgup or Pgdn and/or Home or end.
- With Transfer, Save and Load you can return to continue earlier work another day.

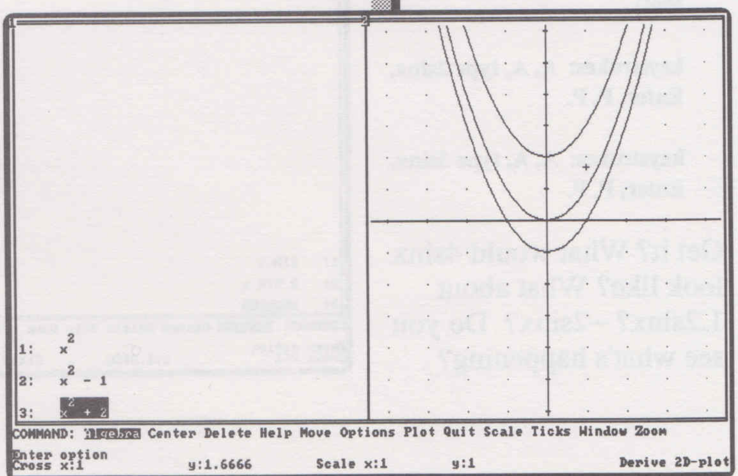
The more you do the more you'll learn. Show your results to others and new ideas will emerge. Sometimes, if a graph is taking too long, with nothing happening on the screen, do a Delete All and start again . . . maybe a look at the scale to suggest F9's or F10's.

The following sequences will take you through many suggestions in a very controlled way. The purpose is to spark your thoughts to action. Each of these sequences is a separate adventure. Clear away the debris before you start a new sequence . . . since your source for the graph is still in the Algebra window, you can always arrow up and plot an earlier expression again.

We have used a monochrome graphics monitor and the HERCULES.COM driver provided with Derive to generate these pictures. Graphs on a CGA monitor will not appear to be the same.

Use the Scale command to adjust the graph or use the F9 key to move in or the F10 key to backup.

keystrokes: A (for Algebra), A (for Author), type x^2, Enter, W (for Window), S (for Split), V (for Vertical), Enter, F1 (to move to the next window),



W (for Window), D (for Designate), 2 (for 2D–plot), y (for yes), P (for Plot).

keystrokes: A, A, type x^2–1, Enter, P, P.

keystrokes: A, A, type x^2+2, Enter, P, P.

Are you predicting the next graph? Try it!

Can you predict the graph of x^2-3? x^2+5? Tap key F10 to back away for a better look.

You might change the y scale to see the same graph from a different view.

keystrokes: S (for Scale), Tab, tap Del to clear, type 10, Enter.

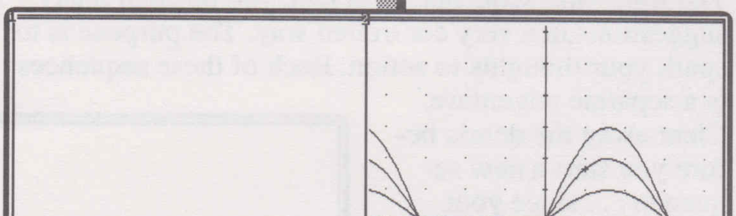
Now change the x scale and see how the graph looks different. Experiment with changes of scale until you are comfortable with scale and use it easily.

keystrokes: A (for Algebra), A (for Author), type sinx, Enter, P (for Plot), P (for Plot).

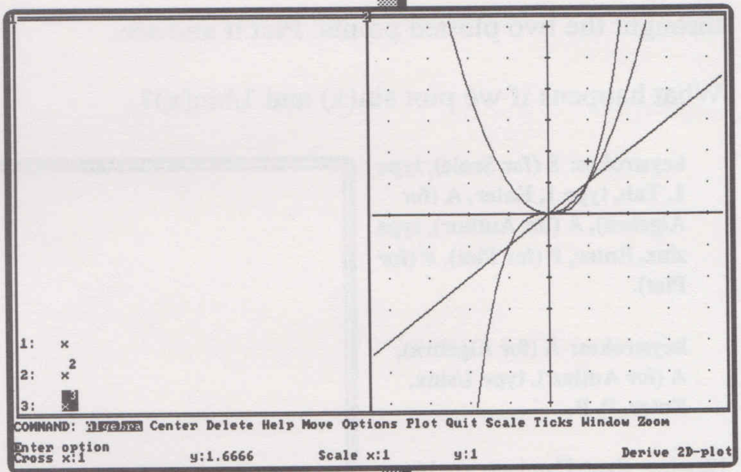
keystrokes: A, A, type 2sinx, Enter, P, P.

keystrokes: A, A, type 3sinx, Enter, P, P.

Get it? What would 4sinx look like? What about 1.2sinx? –2sinx? Do you see what's happening?



keystrokes: D (for Delete), A (for All), A (for Algebra), A (for Author), type x, Enter, P(for Plot), P (for Plot).

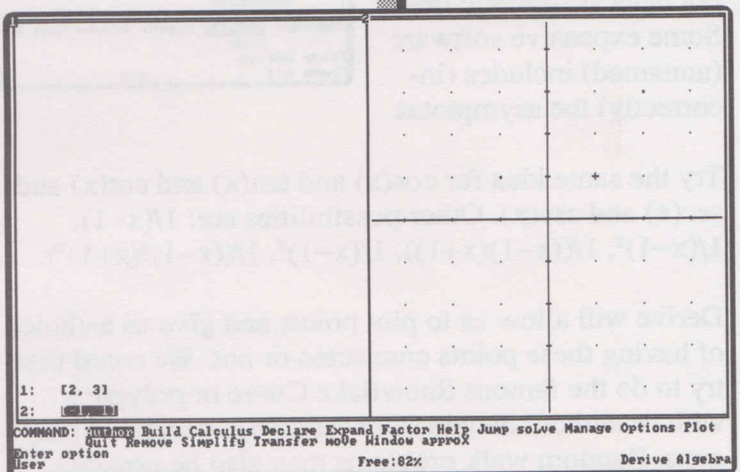


keystrokes: A, A, type x^2, Enter, P, P.

keystrokes: A, A, type x^3, Enter, P, P.

So . . . what would x^4 look like? . . . or x^7? . . . or $x^{(2.5)}$?

Our procedure in this chapter has been to pick a function and then make it's graph. Derive has a command called Fit, which allows us to pick points and then find a function which fits these points.



keystrokes: A (for Algebra), A (for Author), type [2,3], Enter, P (for Plot), P (for Plot), A (for Algebra), A (for Author), type [–1,1], Enter, P (for Plot), P (for Plot).

We'll now find a straight line function which goes through those two points.

keystrokes: A (for Algebra), A (for Author), type Fit [[x, ax+b], [2,3],[–1,1]], Enter, S (for Simplify), Enter.

$$3:\quad \text{FIT}\begin{bmatrix} x & a\,x+b \\ 2 & 3 \\ -1 & 1 \end{bmatrix}$$

$$4:\quad \frac{2\,x}{3}+\frac{5}{3}$$

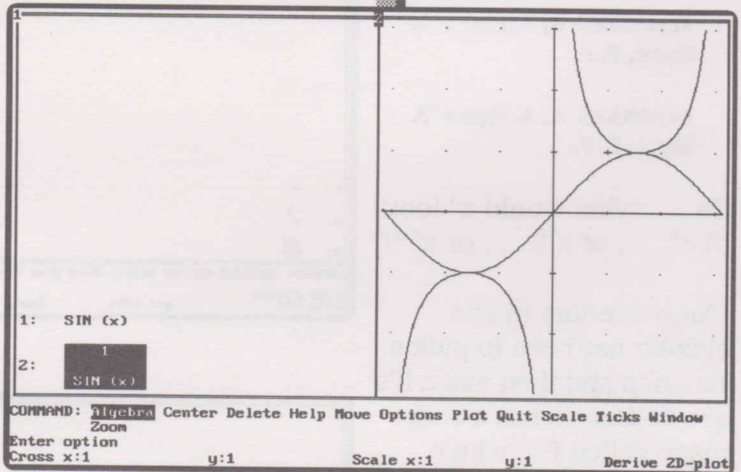
This gives us the straight line function which should go

throught the two plotted points. Plot it and see.

What happens if we plot sin(x) and 1/sin(x)?

> **keystrokes: S (for Scale), type 1, Tab, type 1, Enter, A (for Algebra), A (for Author), type sinx, Enter, P (for Plot), P (for Plot).**

> **keystrokes: A (for Algebra), A (for Author), type 1/sinx, Enter, P, P.**



Notice that Derive, quite correctly, has managed to not have asymptotic lines. Some expensive software (unnamed) includes (incorrectly) the asymptotes.

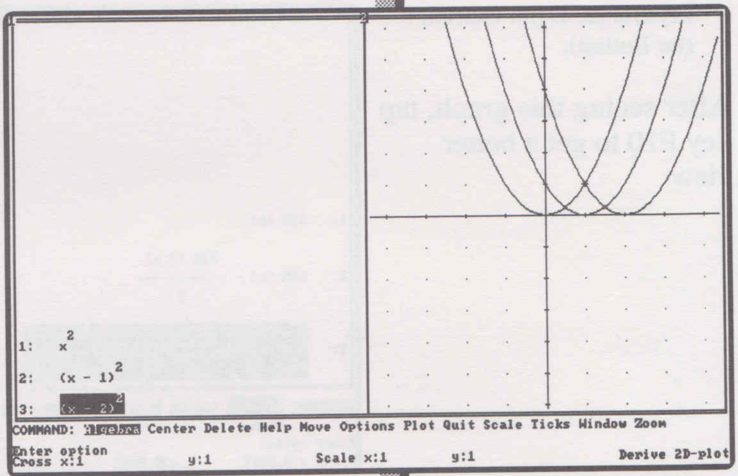
Try the same idea for cos(x) and tan(x) and cot(x) and sec(x) and csc(x). Other possibilities are: $1/(x-1)$, $1/(x-1)^2$, $1/((x-1)(x+1))$, $1/(x-1)^3$, $1/((x-1)^3(x+1)^3)$.

Derive will allow us to plot points and give us a choice of having these points connected or not. We could then try to do the famous Snowflake Curve or polygons with or without diagonals or randomly generated polygons. Random walk problems may also be possible.

keystrokes: A (for Algebra), A (for Author), type x^2, Enter, P (for Plot), P (for Plot).

keystrokes: A, A, type (x–1)^2, Enter, P, P.

keystrokes: A, A, type (x–2)^2, Enter, P, P.



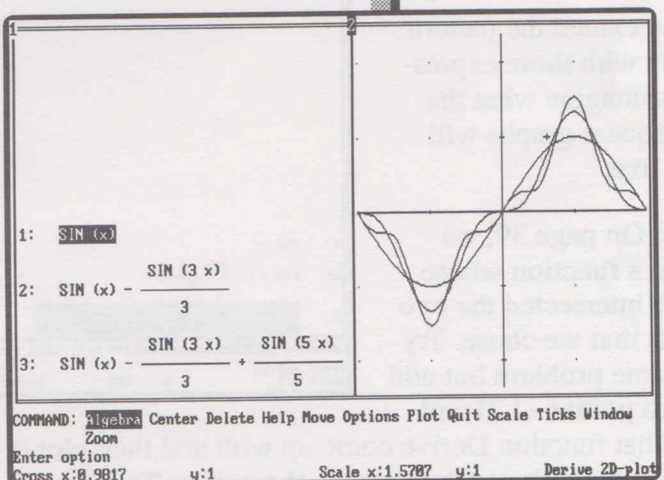
And, if you have the inclination, try $(x-3)^2$ . . . or $(x+3)^2$ or $(x-1.5)^2$.

To clear the screen of previous graphs, tap D (for Delete), A (for All).

keystrokes: A (for Algebra), A (for Author), type sinx, Enter, P (for Plot), P (for Plot).

keystrokes: A, A, type sinx–sin(3x)/3, Enter, P, P.

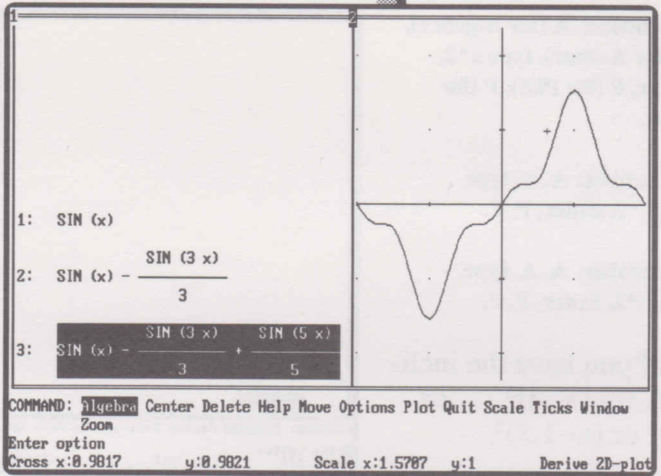
keystrokes: A, A, type sinx–sin(3x)/3+sin(5x)/5, Enter, P, P.



After looking at these graphs, you might like to see only the last graph.

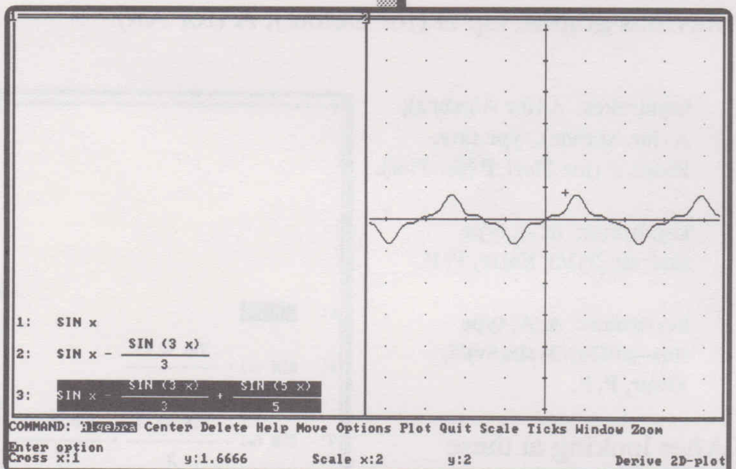
keystrokes: D (for Delete), B (for Butlast).

After seeing this graph, tap key F10 to get a better view.



keystrokes: F10 (to back away).

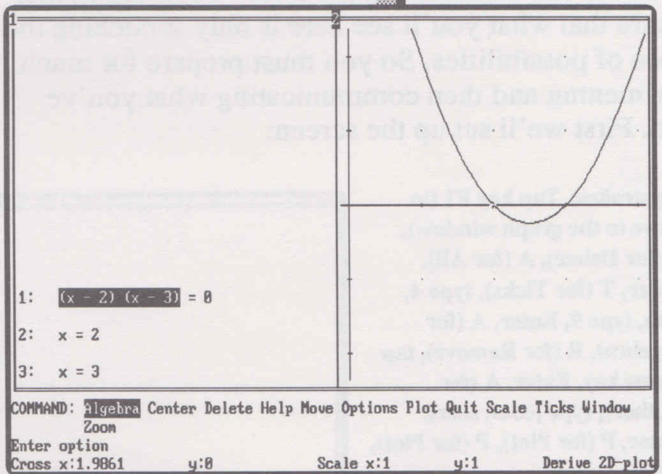
If you extend the pattern begun with these expressions imagine what the subsequent graphs will look like.



Note: On page 39, we found a function whose graph intersected the two points that we chose. Try the same problem but add a third point (–2,2) and see what function Derive come up with and then plot it and see how it's graph relates to the points. Try Fit again but use ax^2+bx+c as the model function and plot the resulting function.

The equation (x–2)(x–3) = 0 has two roots.

keystrokes: A (for Algebra), A (for Author), type (x–2)(x–3)=0, Enter, L (for soLve), Enter.



The results of soLve shows the connection between the equation and the roots. The graph of y=(x–2)(x–3) shows an important connection between the equation and its roots.

keystrokes: Arrowup (to highlight the equation), arrowleft (to highlight just the expression (x–2)(x–3)), P (for Plot), D (for Delete), A (for All), P (for Plot), hold Crtl key down, arrowright three times, C (for Center).

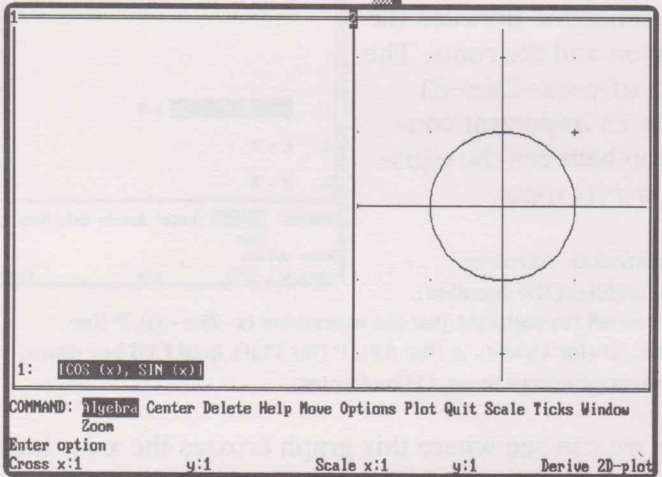
Now we can see where this graph crosses the x (or horizontal) axis. The x values at these two points are the roots of the equation.

Try the same thing for the equation (x–1)(x+1) = 0 and the graph of y = (x–1)(x+1). How is this graph different from the graph of the earlier expression?
Try (x–3)(x+3) = 0 and (x–1)(x–2)(x–3)=0.

Etc., etc., . . . Try the same for cosx . . . predict . . . predict . . . do not worry about being wrong. Worry about not trying enough wrong ideas of your own. Don't forget that Kepler was often wrong . . . same with Euler and all the other big shots in the history of math. If it was OK for them to experiment and make mistakes and learn from their mistakes, it's OK for us, too.

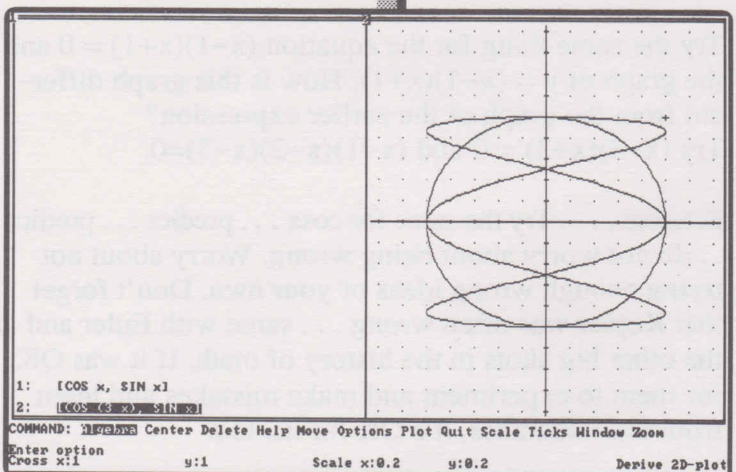
Parametric equations are absolutely my favorite feature on Derive. The possibilities are mind–boggling to me. I'm sure that what you'll see here is only scratching the surface of possibilities. So you must prepare for much experimentng and then communicating what you've found. First we'll set up the screen:

> **keystrokes: Tap key F1 (to move to the graph window), D (for Delete), A (for All), Enter, T (for Ticks), type 4, Tab, type 9, Enter, A (for Algebra), R (for Remove), tap Home key, Enter, A (for Author), type [cosx, sinx], Enter, P (for Plot), P (for Plot), Enter.**



I see such a tiny picture with my scale at one that it wasn't worth all the trouble. A couple of F9's will fix that. . .try it. Wow! A terrific circle (if it does not look like a circle on your screen, experiment with Ticks until it does). Now for a variation.

> **keystrokes: A (for Algebra), A (for Author), type [cos(3x),sinx], Enter, P (for Plot), P (for Plot), Enter.**

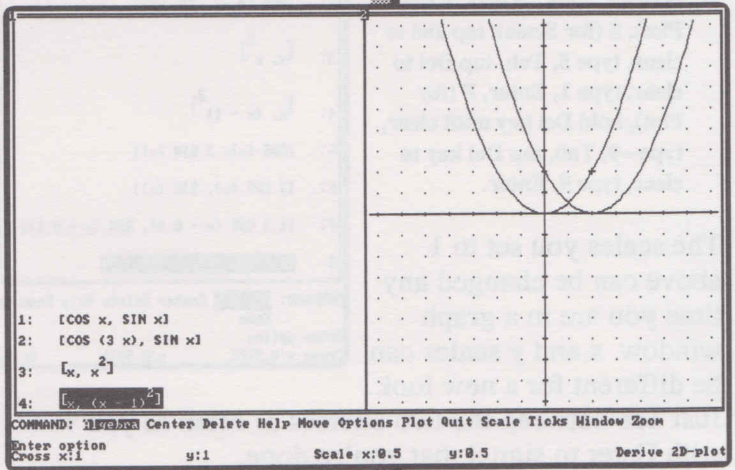


Now you are on your way! A great adventure awaits you. Try [cos(3x), sin(5x)], or [cos(x), sin(3x)], or

[cos(2x), sin(3x)], or. . . go, go, go. . .another direction follows:

keystrokes: D (for Delete), A (for All), A (for Algebra), A (for Author), type [x,x^2], Enter, P (for Plot), P (for Plot), Enter.

keystrokes: A (for Algebra), A (for Author), type [x,(x–1)^2], Enter, P (for Plot), F10 (to back away), P (for Plot), Enter.

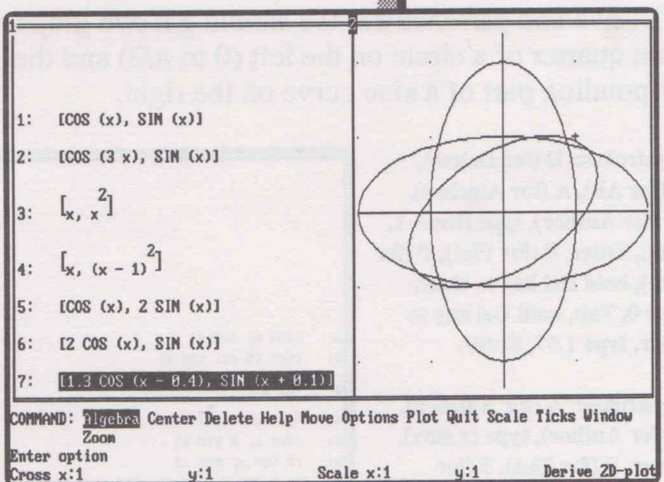


So all the usual graphs you do are possible with parametrics along with new ones.

keystrokes: D (for Delete), A (for All), A (for Algebra), A (for Author), type [cosx, 2sinx], Enter, P (for Plot), P (for Plot), Enter.

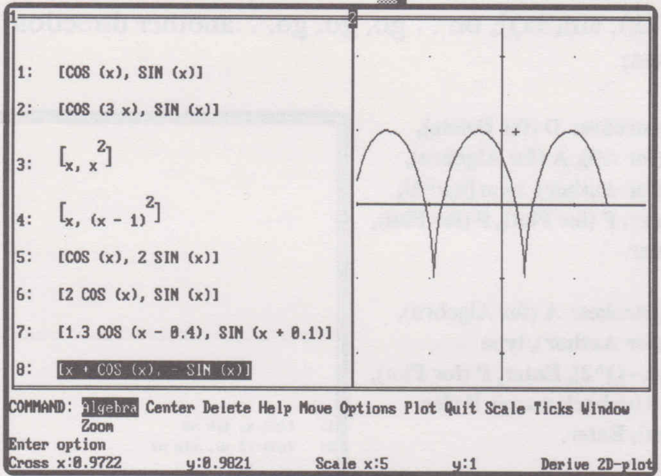
keystrokes: A (for Algebra), A (for Author), type [2cosx, sinx], Enter, P, P, Enter.

keystrokes: A, A, type [1.3cos(x–0.4),sin(x+0.1)], Enter, P, P, Enter.



A cycloid will appear with the next directions. Think of driving along at night and seeing a bicycle in the distance with a reflector in the wheel; the path the reflector takes is often a cycloid. Lots of chances to experiment here.

keystrokes: D (for Delete), A (for All), A (for Algebra), A (for Author), type [x+cosx,–sinx], Enter, P (for Plot), S (for Scale), tap Del to clear, type 5, Tab, tap Del to clear, type 1, Enter, P (for Plot), hold Del key until clear, type –9, Tab, tap Del key to clear, type 9, Enter.

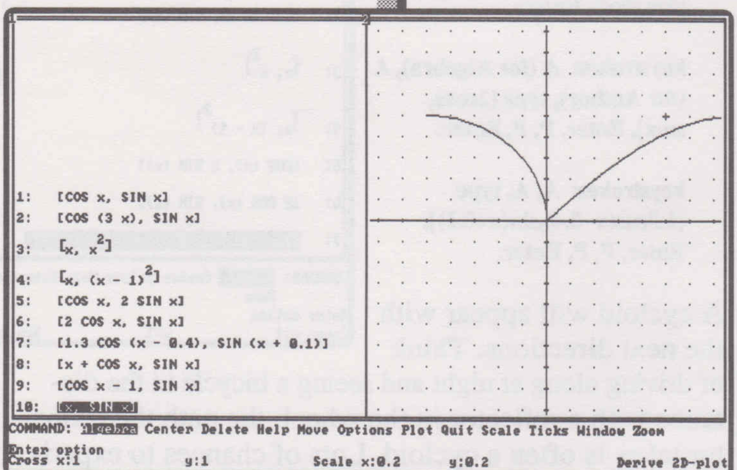


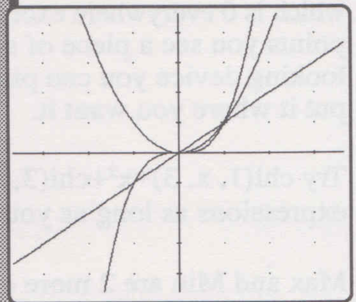
The scales you set to 1 above can be changed any time you are in a graph window. x and y scales can be different for a new look. Just Tab between the two choices as often as you want with Enter to signal that you're done.

This next graph shows the tremendous control you have with Derive and parametrics. We should get two graphs . . . one quarter of a circle on the left (0 to $\pi/2$) and the corresponding part of a sine curve on the right.

keystrokes: D (for Delete), A (for All), A (for Algebra), A (for Author), type [cosx–1, sinx], Enter, P (for Plot), P (for Plot), hold Del key to clear, type 0, Tab, hold Del key to clear, type 1.57, Enter.

keystrokes: A (for Algebra), A (for Author), type [x,sinx], Enter, P (for Plot), S (for Scale), type 0.2, Tab, type 0.2, Enter, P (for Plot), Enter.

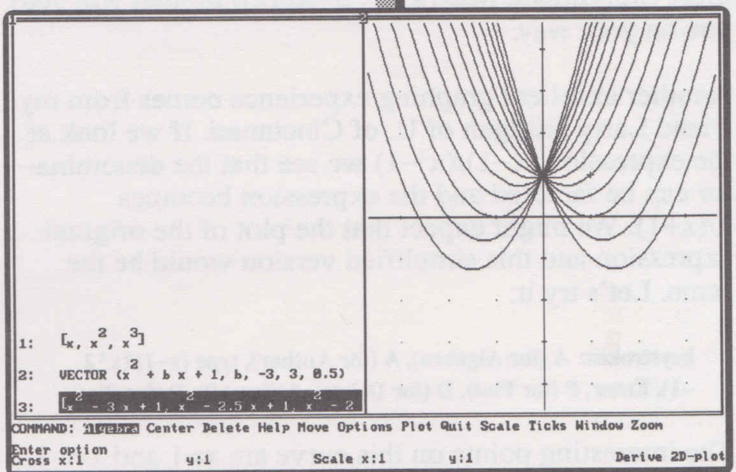




Derive's Plot command will “distribute” across a list. So . . .

keystrokes: A (for Author), type [x, x^2, x^3], Enter, P (for Plot), D (for Delete), A (for All), P (for Plot).

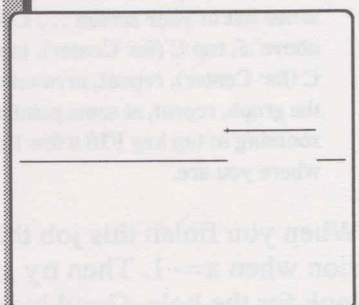
Wonderful effects are possible. Families of curves can easily be generated using the idea above or generating a list of expressions using the Vector command.



keystrokes: A (for Algebra), A (for Author), type vector (x^2–bx+1, b, –3, 3, .5), Enter, X (for approXimate), Enter, P (for Plot), D (for Delete), A (for All), P (for Plot).

Try k*sin(x) or sin(k*x) or sqrt($k-x^2$) or x^2-5x+k with the vector command and let the variable k range across values that you choose.

Another wonderful command for graphing is chi. If I plot chi(1, x, 3) I get a graph which is always 0 except between 1 and 3, when it is 1. I blanked out the axes so you can see the graph.



keystrokes: A (for Algebra), A (for Author),type chi(1, x, 3), Enter, P (for Plot), D (for Delete), A (for All), O (for Options), C (for Color), Enter, Tab, type 0, Enter, P (for Plot).

I thought this was a harmless looking result when I first saw it. Why would anyone want it?

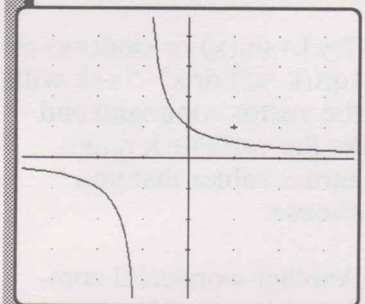
Well, try x^2*chi(1, x, 3) and plot it and you'll see a curve

which is 0 everywhere except between 1 and 3, at which points you see a piece of a parabola. With this harmless looking device you can pick out pieces of any curve and put it where you want it.

Try chi(1, x, 3)*x^2+chi(3, x, 5)*sin(x). String out the expressions as long as you like.

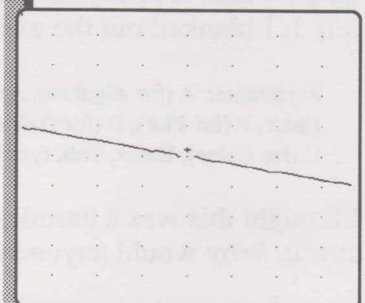
Max and Min are 2 more commands with which you must experiment. Max(x, x^2, 3*sin(x)) plotted will start you on your way.

Another excellent graphing experience comes from my friend Larry Gilligan of U. of Cincinnati. If we look at the expression $(x-1)/(x^2-1)$ we see that the denominator can be factored and the expression becomes $1/(x+1)$. We might expect that the plot of the original expression and this simplified version would be the same. Let's try it:



keystrokes: A (for Algebra), A (for Author), type (x–1)/(x^2 –1), Enter, P (for Plot), D (for Delete), A (for All), P (for Plot).

The interesting points on this curve are x=1 and x=–1. We'll zoom in at the graph when x=1.



keystrokes: Tap the arrowdown key until you reach a point just above the graph, check the x and y value next to Cross at the lower left of your screen . . . x should be 1 and y should be just above .5, tap C (for Center), tap key F9 three times to zoom in, C (for Center), repeat, arrowdown until the Cross is just above the graph, repeat, at some point a gap will appear. If you get lost zooming in tap key F10 a few times to back out and find out where you are.

When you finish this job think about the same expression when x=–1. Then try graphing y=1/(x+1) and look for the hole. Good luck!

Chapter 4

Factoring

Quite often math operations go in at least two directions; up and down or forward and back. Factoring and expanding are back and forward kinds of operations. Here is an example of expanding.

$2(x+3) = 2x+6$

keystrokes: A (for Author), type 2(x+3), Enter, E (for Expand), Enter.

```
1:     2 (x + 3)
2:     2 x + 6
```

Here is the corresponding factoring.

$2x+6 = 2(x+3)$

keystrokes: F (for Factor), Enter, R (for Rational).

```
3:     2 (x + 3)
```

Here is expanding.

$7(3x+2) = 21x+14$

keystrokes: A (for Author), type 7(3x+2), Enter, E (for Expand), Enter.

```
4:     7 (3 x + 2)
5:     21 x + 14
```

And here is the corresponding factoring:

keystrokes: F (for Factor), Enter, R (for Rational).

```
6:     7 (3 x + 2)
```

Here are some factoring opportunities for you to try. See if you can guess the results before Derive does it.

Opportunities:

$25x+50$	$36+9x$	$27x-33$
$2x^2+12x+14$	$100z^2+75z+25$	$121x^6+11$
$y^5+y^4+y^3$	$x^3y^2z-xy^2z^3$	x^5-x^2

One way to gain skill in factoring is to guess, record the result, and then ask Derive to perform the same operation so that you can check your work. Derive lets you record a guess; it doesn't "comment" on your guess.

keystrokes: A (for Author), type 7x+21z–35y=7(x+3z–5y), Enter.

```
7:    7 x + 21 z - 35 y = 7 (x + 3 z - 5 y)
```

We want Derive to factor the left side of the equation, which in effect is the problem, and we want Derive to leave the right side of the equation (our guess) alone. It can be done.

keystrokes: Arrowleft (to highlight 7x+21z–35y), F (for Factor), Enter, Enter, T (for Trivial).

```
8:    7 (x - 5 y + 3 z) = 7 (x + 3 z - 5 y)
```

The word Trivial here means factor out a letter or number which is common to each term.

Use this approach when you want to record a guess for factoring. Derive does not judge your guess, but gives you what you need to judge for yourself.

Some expressions contain more than one instance of a variable. These, too, can be factored:

keystrokes: A (for Author), type 7x^4+5x^3–3x^2, Enter, F (for Factor), Enter, T (for Trivial).

$$9: \quad 7x^4 + 5x^3 - 3x^2$$

$$10: \quad x^2 (7x^2 + 5x - 3)$$

Sometimes numbers and variables can both be factored out.

keystrokes: A (for Author), type 11x^7+33x^3–22x^2, Enter, F (for Factor), Enter, R (for Rational).

$$11: \quad 11x^7 + 33x^3 - 22x^2$$

$$12: \quad 11x^2 (x^5 + 3x - 2)$$

Sometimes the world can become complicated:

$28x^4y^2-7xy+21x^3y^2$

Try these:

$45x^3z^7-18x^2yz$ Guess when you're ready!
$12a^3b^3c^3+24abc$ Make up your own!!

```
13:  28 x^4 y^2 - 7 x y + 21 x^3 y^2
14:  7 x y (4 x^3 y + 3 x^2 y - 1)
```

In algebra classes in school a restricted kind of factoring is developed. With Derive we can do all of this factoring plus quite a bit beyond. We also can approach the topic by looking for patterns and not just following rules that are learned from books or teachers. The results are similar.

$x^2-9 = (x-3)(x+3)$

keystrokes: A (for Author), type x^2–9, Enter, F (for Factor), Enter, R (for Rational).

```
15:  x^2 - 9
16:  (x - 3) (x + 3)
```

x^2-9 is called a binomial because it has two terms. Both x^2 and 9 are perfect squares since $x*x = x^2$ and $3*3 = 9$. So . . . x^2-9 is called the difference of 2 squares. Below are some more differences of 2 squares. Guess what they factor to . . . try it:

Opportunities:

x^2-25	x^2-64	y^2-16	z^2-49	a^2-225
x^2-144	z^2-400	y^2-81	a^4-625	

Make your own like these!

It is possible to complicate the issue with some new examples.

$4x^2-y^2$ $25y^2-16x^2$ $100z^2-9x^2$ $441z^4-225x^2$

Another kind of factoring involves trinomials; that is expressions with 3 terms. In these examples we are also working with quadratic expressions . . . meaning that the highest power is 2.

keystrokes: A (for Author), type x^2+5x+6, Enter, F (for Factor), Enter, R (for Rational).

17: $x^2 + 5x + 6$

18: $(x + 2)(x + 3)$

$x^2+12x+35 = (x+5)(x+7)$

19: $x^2 + 12x + 35$

20: $(x + 5)(x + 7)$

Do you see a pattern? Look at the 12 and 35 and then the 5 and 7.

$x^2+10x+21 = ?$

21: $x^2 + 10x + 21$

22: $(x + 3)(x + 7)$

If you see patterns, you can make up a trinomial that will factor nicely like these. Random trinomials we make up will probably not factor, especially if we stay with Factor–Rational. You have to see patterns in the above examples to be able to make new ones.

$x^2 - 7x + 5$ is a trinomial that does not have whole number factors:

keystrokes: A (for Author), type x^2–7x+5, Enter, F (for Factor), Enter, R (for Rational).

$x^2 - 7x + 5$

$x^2 - 7x + 5$

As you can see, when Derive cannot factor an expression at the level asked for, it returns the same expression. This last expression was the first in this chapter that did not respond to the Factor–Rational treatment, so now we'll go to stronger medicine.

keystrokes: F (for Factor), Enter, D (for raDicals).

1: $x^2 - 7x + 5$

2: $\left[x - \frac{\sqrt{29} + 7}{2}\right]\left[x + \frac{\sqrt{29} - 7}{2}\right]$

This result raises an interesting question: What relationship exists between the -7 and 5 in x^2-7x+5 and the numbers in the factored form? There are a number of experiments to try, if you are interested. You might like to see the factored expression in a different form. Here again, Derive can help.

keystrokes: Arrowup (to highlight x^2-7x+5), O (for Options), P (for Precision), A (for Approximate), Enter, F (for Factor), Enter, D (for raDical).

1: $x^2 - 7x + 5$

2: $\left[x - \dfrac{\sqrt{29} + 7}{2}\right] \left[x + \dfrac{\sqrt{29} - 7}{2}\right]$

3: $(x - 6.19258)\ (x - 0.807417)$

Do O (for Options), P (for Precision), E (for Exact), Enter, since Exact mode is where you usually want to be.

Here are some examples of "nice" trinomials that will factor and might help your thinking.

$y^2-7y+12$	$z^2-12z+35$	$x^2+18x+77$
$x^2+14x+45$	$16x^2+40x+25$	$4z^2+20z+25$
$9x^2+10x+1$	$5x^2+11x+2$	$x^2+21x+20$
$7x^2+22x+3$	$7x^2+10x+3$	$3x^2+16x+5$

Make up your own examples of binomials and trinomials that you think will factor. If Derive's results are different from yours, you may both be right. Check by expanding your results and Derive's results.

Look in math textbooks for more problems to do.

A student of mine, Usama, when he was in the second grade, became interested in factoring. This unusual interest was aroused by watching older students working on the topic using Derive and another symbolic algebra program called Mathematica. Try Usama's investigation yourself and see what results you can predict.

Usama's expressions: x^2-1 x^3-1 x^4-1 and so on.

keystroke: A (for Author), type x^2–1, Enter, F (for Factor), Enter, R (for Rational).

keystroke: A, type x^3–1, Enter, F, Enter, R.

```
          2
23:  x  - 1
24:  (x - 1) (x + 1)
          3
25:  x  - 1
                  2
26:  (x - 1) (x  + x + 1)
```

Within Usama's collection of problems there are patterns, for example: x^2-1, x^4-1, x^8-1, $x^{16}-1$. Also look at x^2-1, x^3-1, x^5-1, x^7-1, $x^{11}-1$, $x^{13}-1$. What's next in this sequence? Usama was not thinking of $x^{15}-1$. Also consider x^5-1, $x^{25}-1$, $x^{125}-1$, . . . There are many streets to explore. This is where the real strength of Derive can show.

Guess, guess, guess! Don't worry about being wrong. The only result better than one wrong guess is two wrong guesses. The better the math student, the higher the error rate. Only the poor math student is afraid to guess.

Perfect Square Trinomials

Some trinomials factor into two identical factors, so these trinomials are called perfect squares. Try these and make up your own:

$x^2+10x+25$	$y^2-14y+49$	$4x^2+36x+81$
$121x^4-22x^2+1$	$z^2-42z+441$	$9x^2-60+100$

Completing the Square

If we start with x^2-6x+9 and factor . . . we get the perfect square $(x-3)^2$.

```
          2
27:  x  - 6 x + 9
              2
28:  (x - 3)
```

$29: \; x^2 - 6x + 7$

$30: \; x^2 - 6x + 7$

If we start with $x^2–6x+7$ and factor, Derive gives us back the same thing. This means it can't be factored with whole numbers. Sometimes, out of necessity, math people decide to change an expression like $x^2–6x+7$ into something like a perfect square. How do we make $x^2–6x+7$ into a perfect square? What number do we need instead of 7 to make this a perfect square? Since $(x–2)^2 = x^2–4x+4$ and $(x–3)^2 = x^2–6x+9$, we have, as usual, some choices to make. The 6x matching makes $(x–3)^2$ attractive.

$x^2–6x+7 = x^2–6x+9–2 = (x–3)^2–2.$

$31: \; x^2 - 6x + 9 - 2$

$32: \; x^2 - 6x + 9$

$33: \; (x - 3)^2$

$34: \; x^2 - 6x + 7 = (x - 3)^2 - 2$

$35: \; x^2 - 6x + 7 = x^2 - 6x + 7$

To check, we expand. Try to "complete the square" for each of the following:

$x^2–8x+10$ $\quad x^2+20x+7$ $\quad x^2–14x+46$ $\quad y^2–10x+24$

Opportunities:

Factor: $x^2–12x+36$ $\quad x^2–12x+35$ $\quad x^2–12x+32$

$x^2–12x+27$ $\quad x^2–12x+20$ $\quad x^2–12x+11$

See the patterns?? Graph each of these!! See?? Make up more like this and predict the relationship to the graphs.

To set up a 2D graphing window:

keystrokes: W (for Window), S (for Split), V (for Vertical), Enter, F1 (to move to the next window), W (for Window), D (for Designate), 2 (for 2–D Plot), y (for yes), O (for Options), D (for Display), G (for Graphics), Enter.

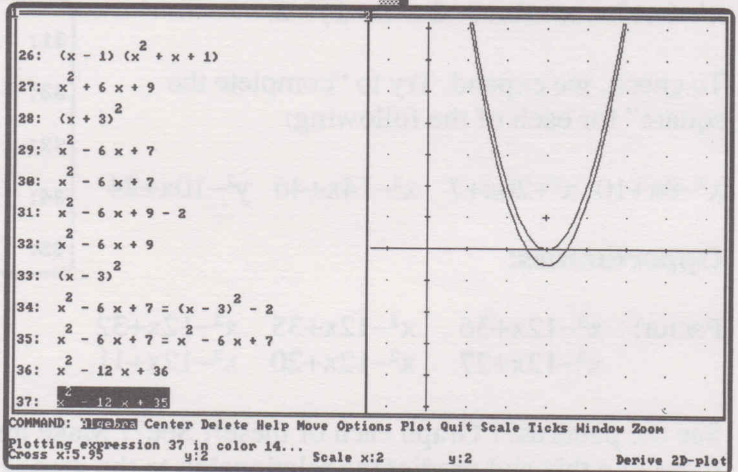
This will create a graphing window.

keystrokes: A (for Algebra), A (for Author), type x^2–12x+36, Enter, P (for Plot), S (for Scale), tap Del to clear, type 2, Tab, tap Del to clear, type 2, Enter, hold Crtl key and tap arrowright 4 times, C (for Center), P (for Plot).

keystrokes: A, A, tap key F3 (to copy the highlighted expression), tap backspace key once, type 5, Enter, P, P.

Look where these graphs cross the x or horizontal axis and locate the expression they come from. Your graphs may be seen more easily with the following keystrokes:

keystrokes: Use arrow keys to move the marker (+) on the graph window to the bottom of the first graph, tap C (for Center), tap F9 to move closer to the graphs, use the arrow keys to move the marker (+) to crossing points and read the x and y coordinates from the lower left of your screen.



If you see nothing in the graphing window, press the F10 key once or twice to back away from the graphing field (I think of it as a field I'm flying over in a helicopter); and F9 moves me lower toward the field and F10 moves me away from the field of graphs.

The expression $x^2–5x+7$ gives us a chance to look at some interesting possibilities with factoring. Before factoring I'll give some keystrokes to set the form of the answer.

keystrokes: O (for Options), P (for Precision), E (for Exact), Enter, A (for Algebra), A (for Author), type x^2–5x+7, Enter, F (for Factor), Enter, R (for Rational), F (for Factor), Enter, D (for raDicals), F (for Factor), Enter, C (for Complex).

```
38:  x^2 - 5 x + 7
39:  x^2 - 5 x + 7
40:  x^2 - 5 x + 7
```

$$41: \left[x - \frac{5 + \sqrt{3}\,\hat{\imath}}{2}\right]\left[x + \frac{-5 + \sqrt{3}\,\hat{\imath}}{2}\right]$$

As you see, Factor–Rational had no effect, Factor–raDicals had no effect, but Factor–Complex did it. We have just met the symbol î, one of the units in complex numbers (î is for "imaginary") which will open up a vast area to explore. With the following keystrokes we can reach another useful form.

keystrokes: Arrowleft (to highlight x–(5+sqrt3î)/2)), X (for approXimate), Enter, arrowright twice (to highlight x+(–5+sqrt3î)/2), X (for approXimate), Enter.

$$41: \left[x - \frac{5 + \sqrt{3}\,\hat{\imath}}{2}\right]\left[x + \frac{-5 + \sqrt{3}\,\hat{\imath}}{2}\right]$$

$$42: (x - 2.5 - 0.866025\,\hat{\imath})\left[x + \frac{-5 + \sqrt{3}\,\hat{\imath}}{2}\right]$$

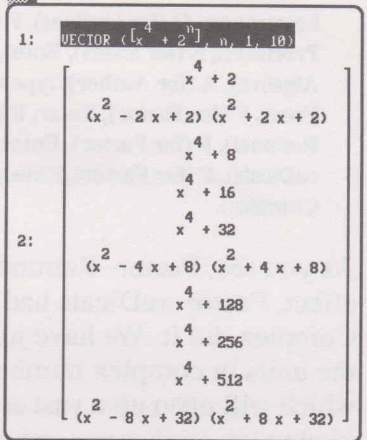
$$43: (x - 2.5 - 0.866025\,\hat{\imath})\,(x - 2.5 + 0.866025\,\hat{\imath})$$

Almost exactly the same result can be reached by highlighting x^2–5x+7, doing Options–Precision, Mixed and then Factoring.

Develop the habit of trying to get the form you want from Derive. I can't guarantee you'll always get just what you want (I don't); but, I am sure that with experience, your results will be amazingly good.

At a recent meeting at Joliet Community College, a teacher and I were both surprised when Derive factored x^4+64 with Rationals. We intuitively expected an echo, meaning no action, and thought we'd go on to factor across the raDicals or Complex. This opening led us to investigate similar forms.

keystrokes: A (for Author), type vector([x^4+2^n], n, 1, 10), Enter, F (for Factor), Enter, Enter, R (for Rational), Enter.



This gave us some clues and we then tried a table of $x^4+2^{(4n-2)}$. See what you can find and extend this opening as far as you can go.

Another factoring problem I learned first from Wally Dodge (who I later heard got it from Charlie Schultz - two excellent math teachers at New Trier High School, near Chicago). If we factor x^2-1 or $x^{12}-1$ and look for patterns, we hardly notice that all the coefficients of the factors are 0 or 1 or -1. We don't notice because they all come out that way. Try a few and see. Then try factoring $x^{105}-1$ with the Rationals and look closely at the factors. Tap arrowright to isolate the first factor and tap arrowright to highlight the second factor; keep going. If the new factor is longer than one line then hold down the Ctrl key and tap the arrowright to bring on the rest of the long factor. When you've seen all of the long factor, release the Ctrl key and tap arrowright for the next factor. You should find that the last 2 factors have 1 term each with a non 0 or 1 or -1 coefficient. 105 is the lowest n of x^n-1 to do this. These expressions are called Cyclotomic Polynomials and now you are on your way.

Chapter 5

Sums

1+2+3+4 = x is a simple statement. What number for x will make it true?

1+2+3+ . . . +98+99+100 = x is also a simple statement but it involves many more numbers than our first statement above. What number for x makes it true? (By the way, this statement means "add up the numbers from 1 to 100").

We need a strategy. Try taking the harder problem and skipping it; just don't do it. It's often the best thing to do with hard problems. Instead, we'll look at the easier problem, but not in the usual way, in a rather clever way.

1+2+3+4
5
5

So, 1+2+3+4 = 5+5 = 2*5 = 10. (Do you see what we're doing?) Let's try a slightly harder problem of the same type:

1+2+3+4+5+6
7
7
7

1+2+3+4+5+6 = 7+7+7 = 3*7 = 21

Where is the 3 from? Where is the 7 from?

Try more problems like these. Use Derive to check your answers.

One way to do 1+2+3+4 on Derive:

keystrokes: A (for Author), type n, Enter.

keystrokes: C (for Calculus), S (for Sum), Enter, Enter, Tab, tap Del to clear, type 4, Enter.

keystrokes: S (for Simplify), Enter.

```
1:    n

        4
2:      Σ  n
       n=1

3:    10
```

You should get 10 as a result.

The symbol $\sum_{n=1}^{4} n$ is read "the sum of some number n from n=1 to n=4 "and means 1+2+3+4.

To do 1+2+3+4+5+6:

keystrokes: A (for Author), type n, Enter.

keystrokes: C (for Calculus), S (for Sum), Enter, Enter, Tab, tap Del to clear, type 6, Enter.

keystrokes: S (for Simplify), Enter.

```
4:    n

        6
5:      Σ  n
       n=1

6:    21
```

You should get 21 as a result.

Now we can return to the harder of our original problems.

1+2+3+ . . . +98+99+100 = x.

We can use Derive to answer the question, what number for x will make this statement true?

keystrokes: A (for Author), type n, Enter.

keystrokes: C (for Calculus), S (for Sum), Enter, Enter, Tab, tap Del to clear, type 100, Enter.

keystrokes: S (for Simplify), Enter.

```
7:    n

       100
8:      Σ  n
       n=1

9:    5050
```

Isn't this the power of the computer? By typing about the same number of keys you get the answer for 1+2+3+4, an easy problem, or for 1+2+3+ . . . +98+99+100, a harder problem. It makes you change your mind about what is a hard problem and what is an easy problem. Computers do have that effect on people.

Now that we see the strength of the computer, how about seeing the strength of our minds? Go back to our problems and let's analyze further (this is where the computer is presently weaker).

$1+2+3+4 = 5+5 = 2*5.$

We see two numbers in 2*5; the 5 is from the 1+4 or the 2+3, but where is the 2 from? How many pairs of 5's can you make from 1+2+3+4? Yes, 2 pairs from 4 numbers.

$1+2+3+4+5+6 = 7+7+7 = 3*7.$

So the 7 is from 6+1 and the 3 might be the number of pairs you can make from 6 numbers.

$1+2+3+ \ldots +98+99+100 = 5050$ answer from Derive.

Could we get the answer to this "difficult problem" without consulting the computer?

Well . . . certainly 101 is 1+100 and 2+99 and so on. So 101 must play a big part in our answer. How many 101's do we have? From 1 to 100 is 100 numbers so there must be 50 pairs of numbers. So . . .

$1+2+3+ \ldots +98+99+100 = 50*101.$

How does that answer compare with Derive's answer?

Opportunities:

Get the sum from 1 to 10, 1 to 20, 1 to 30, 1 to 40 and so on. Look at the answers and search for patterns. Predict future answers and check your results. This is a good place to use again some of Derive's features to make our lives easier.

In this last sequence of sums from 1 to 10 and so on you could type in each problem separately but there is a better way. Try this sequence to get the first three in this collection:

keystrokes: A (for Author), type n, Enter.

keystrokes: C (for Calculus), S (for Sum), Enter, Enter, Tab, tap Del to clear, type 10, Enter.

keystrokes: S (for Simplify), Enter.

10: n

11: $\sum_{n=1}^{10} n$

12: 55

This is our normal sequence for sums. Here comes the effort–saving editing technique:

keystrokes: Arrowup (to highlight $\sum_{n=1}^{10} n$), A (for Author), tap key F3 (to copy the highlighted expression), hold the Ctrl key down and tap A twice, and let up on the Ctrl, type 2, Enter.

keystrokes: S (for Simplify), Enter.

13: $\sum_{n=1}^{20} n$

14: 210

keystrokes: Arrowup (to highlight $\sum_{n=1}^{20} n$), A (for Author), F3 (to copy), hold Ctrl down and tap A twice and let up on Ctrl, type 3, Enter.

keystrokes: S (for Simplify), Enter.

15: $\sum_{n=1}^{30} n$

16: 465

Use this editing technique often and you will learn to like it . . . especially on long statements. Thanks again

to Al Rich and David Stoutemyer for excellent design and programming in Derive.

Opportunities:

1. Find the sum of n^3 instead of n. Do it from 1 to 2 (that is, 1^3+2^3), from 1 to 3, $(1^3+2^3+3^3)$ and so on. Using Derive, the only change from finding the sum of the n numbers is that you type n^3 instead of n at the author mode. Look for patterns . . . you'll find them.

2. Find the sum of n^2 (harder pattern to see than n^3).

3. Find the sum of the first 5 even numbers (2n representing the even numbers and n going from 1 to 5).

4. Find the sum of $1/2^n$ with n going from 1 to 2, which results in 1/2+1/4, and n going from 1 to 3, resulting in 1/2+1/4+1/8, and so on.

5. Find the sum of $1/3^n$, $1/4^n$, and so on. There are wonderful patterns in the answers of the separate sequences and overall. Find them with Derive's help.

Derive gives 7/8 as the sum from 1 to 3 of $1/2^n$. This shows a special quality of Derive. Most programs produce answers only in decimal form; Derive gives you a choice. You may specify a rational number (fraction or ratio) or a decimal. Mathematically, each can be useful.

For example, in the case of $1/2^n$, if n goes from 1 to 10 the result in fraction form, 1023/1024, shows very clearly what is happening (try it from 1 to 20 and it will be even clearer). But when you try the same approach on $1/4^n$ the decimal form (type X for approXimate) may show you a clearer pattern than the fraction form does.

6. Find the sum of 1/n with n going from 2 to 3, which results in 1/2+1/3.

Find the sum of 1/n with n going from 2 to 5, which results in 1/2 + 1/3 + 1/4 + 1/5.

Find the sum of 1/n with n going from 2 to 1000, which results in 1/2 + 1/3 + . . . + 1/999 + 1/1000.

On my little laptop Toshiba T–1000 (a 4.77 megahertz computer) it took the time to get coffee to produce a many–digit, exact, rational number. When I pressed X for approXimate to change 1/2 + 1/3 + . . . + 1/999 + 1/1000 to a decimal, the result was 6.48547 with precision set to 6 digits and 6.48547086566 when set to 12 digits.

Let me show you how to get Derive to do the 1/n sum:

keystrokes: A (for Author), type 1/n, Enter.

keystrokes: C (for Calculus), S (for Sum), Enter, Enter, tap Del to clear, type 2, Tab, tap Del to clear, type 100, Enter.

keystrokes: X (for approXimate), Enter.

17: $\frac{1}{n}$

18: $\sum_{n=2}^{100} \frac{1}{n}$

19: 4.18737

The sum of the numbers in the form 1/n is a famous sequence called The Harmonic Sequence. It was proved long ago that the sum increases without bound as n increases. We found on Derive that the first 1000 terms adds to about 6 1/2; the first 10000 terms adds to about 9.8.

Derive has much more power in this area than we have shown. It can generate symbolic results as well as numerical results for sums.

keystrokes: A (for Author), type n, Enter, C (for Calculus), S (for Sum), Enter, Enter, Tab, tap Del to clear, type k, Enter, S (for Simplify), Enter.

17: n

18: $\sum_{n=1}^{k} n$

19: $\dfrac{k\,(k+1)}{2}$

. . . or do something that seems harder . . .

keystrokes: A (for Author), type n^7, Enter, C (for Calculus), S (for Sum), Enter, Enter, Tab, tap Del to clear, type k, Enter, S (for Simplify), Enter.

20: n^7

21: $\sum_{n=1}^{k} n^7$

22: $\dfrac{k^2\,(k+1)^2\,(3\,k^4+6\,k^3-k^2-4\,k+2)}{24}$

If you don't like this form, try this change:

keystrokes: E (for Expand), Enter.

23: $\dfrac{k^8}{8}+\dfrac{k^7}{2}+\dfrac{7\,k^6}{12}-\dfrac{7\,k^4}{24}+\dfrac{k^2}{12}$

And while we're at it, let's show off by factoring that ugly expression.

keystrokes: F (for Factor), Enter, R (for Rational).

24: $\dfrac{k^2\,(k+1)^2\,(3\,k^4+6\,k^3-k^2-4\,k+2)}{24}$

Since statement 22 and 24 came out the same you might think that Derive produced statement 24 by remembering statement 22. I swear, that is not the case. On my little six pound 4.77 megahertz Toshiba T1000 laptop, Derive worked and worked for 41 seconds to get the result. On my 20 megahertz 386 notebook it took 4.59 seconds and on my 33 megahertz 486 it took 1.48 seconds.

Notes:

Chapter 6

Solving Linear Equations

Many linear equations are easily solved.

$2x+3 = 11$

This equation states that 2 times some number plus 3 may equal 11. What number for x can make this open statement true? Let's try a number for x. If it makes the open sentence true, we're done; if it makes the open sentence false, we try a larger or smaller number until one number works. In this case, 4 works because $(2*4)+3 = 11$ is true.

Now look at $127x-315+211-36x+2x = 600+5x$.

This is also a linear equation and I will resist all temptations to solve it by myself, because it is a bit messy. Here is where Derive can be a big help. We'll try to solve this equation by inspection but with some help from Derive.

keystrokes: A (for Author), type 127x–315+211–36x+2x=600+5x, Enter.

```
1:    127 x - 315 + 211 - 36 x + 2 x = 600 + 5 x
```

keystrokes: M (for Manage), S (for Substitute), Enter, type 3, Enter.

```
2:    127 3 - 315 + 211 - 36 3 + 2 3 = 600 + 5 3
```

Here we've told Derive to substitute 3 for x (this is just a wild guess on my part) but not to carry out the calculation yet. This is a big help to see and understand what's going on; it slows the computer for us. (Thanks to Al Rich and David Stoutemyer for Derive's excellent design and programming!) Now we tell Derive to go to it and do the messy part we want to skip.

keystrokes: S (for Simplify), Enter.

```
3:    175 = 615
```

So we learn that my guess of 3 was too big or too small and does not make the open sentence true. I'd have been shocked if it had worked right off. My next wild guess is 10.

keystrokes: arrowup twice (to highlight the original equation), M (for Manage), S (for Substitute), Enter, type 10, Enter.

```
4:    127 10 - 315 + 211 - 36 10 + 2 10 = 600 + 5 10
```

This told Derive to substitute 10 in for every x. Now we'll have Derive do the messy part:

keystrokes: S (for Simplify), Enter.

```
5:    826 = 650
```

Let's see . . .
. . . when $x = 3$ is substituted, we get $175 = 615$.

. . . when $x = 10$ is substituted, we get $826 = 650$.

I think 10 is too big since 826 is larger than 650. (Can you see why I come to this conclusion?)

Next I'll try 8. (What would you try?) If you have Derive in your computer and running (you should) then follow this up yourself before you read ahead.

Trying 8 the same way as above, I find it works. (Lucky guess!) Allowing Derive do the messy bits, we have solved an arithmetically complicated linear equation by inspection. Try this technique with all equations you meet for a while. You'll learn a lot from this experience. There are many interesting and delicious surprises.

When you need an answer, Derive will often get it for you. But if you also want to do some mathematical thinking, use these commands: manage, substitute and simplify.

keystrokes: Arrowup (to highlight the original equation, 127x–315+211–36x+2x=600+5x), S (for Simplify), Enter.

```
7:   93 x - 104 = 5 x + 600
```

What has happened? The right side of the equation looks about the same but the left side has changed. Derive has simplified 127x–36x+2x into 93x and simplified –315+211 into –104. Now comes our job. We'd like only one side of the equal sign to have x's and the other side to have nothing but numbers. First the x's.

keystrokes: B (for Build), Enter, type –, arrowright twice (to highlight 5x+600), arrowdown (to highlight 5x), Enter, D (for Done).

```
9:   (93 x - 104 = 5 x + 600) - 5 x
```

This directs Derive to subtract 5x from both sides of the equation.

keystrokes: S (for Simplify), Enter.

```
10:  88 x - 104 = 600
```

This makes Derive carry out the indicated subtraction. Next we must add 104 to both sides of the equation.

keystrokes: B (for Build), Enter, type +, arrowleft (to highlight 88x–104), arrowdown (which highlights 88x), now arrowright (to highlight 104), Enter, D (for Done).

```
12:  (88 x - 104 = 600) + 104
```

And again we direct Derive to carry out the indicated operation.

keystrokes: S (for Simplify), Enter.

```
13:  88 x = 704
```

Now we can divide both sides of the equation by 88.

keystrokes: B (for Build), Enter, type /, arrowleft (to highlight 88x), arrowdown (to highlight 88), Enter, D (for Done).

$$15:\quad \frac{88\ x = 704}{88}$$

And to simplify . . .

keystrokes: S (for Simplify), Enter.

```
16:  x = 8
```

Of course, for those who are interested only in the answer, Derive can be used in non slow–motion mode.

keystrokes: Arrowup (to highlight the original equation), L (for soLve), Enter.

```
17:  x = 8
```

All linear equations can be solved by each of these three methods:

Method 1: Guess a number and use Manage and Substitute to see if it makes the equation true. Let Derive calculate the results, and use the feedback to make a new and improved guess. Keep going until you find a solution (or get tired!).

Method 2: By adding or subtracting or multiplying or dividing, transform the original equation into one that you can solve by inspection (the Build command is most useful).

Method 3: Use the soLve command.

Try all three methods on this equation:

$7x+22-6x+3-5236 = -13x+889+64x.$

Don't let anyone easily convince you–in any part of math–that there is only one way to do it. The person may know only one way or may like only one way, but rarely is there only one way to solve a problem. I've described three here. Find another way. Send your solution to us. It may appear in the next edition of this book, with our acknowledgment that you are the one who found it.

A wonderful feature of Derive allows us to solve more than one linear equation at a time. I'll make an example; you can do your own. I'll imagine two numbers, x and y, and their sum is 27, and their difference is 5. What numbers will work?

keystrokes: A (for Author), type [x+y = 27, x–y = 5], Enter, L (for soLve), Enter.

```
1:   [x + y = 27, x - y = 5]
2:   [x = 16, y = 11]
```

To check these results:

keystrokes: Arrowup, M (for Manage), S (for Substitute), Enter, type 16, Enter, type 11, Enter, S (for Simplify), Enter.

```
3:   [16 + 11 = 27, 16 - 11 = 5]
4:   [27 = 27, 5 = 5]
```

In this case we had numerical solutions: try the same approach but with equations $x+y = 27$ and $2x+2y = 54$.

The @ symbol means any number can be substituted for the variable and therefore the two equations are really the same equation written in a different form and are called "not independent". If you graph both of them you'll see only 1 line.

You should also try the same approach with x+y = 27 and x+y = 28. Their graphs should be parallel and therefore no solution exists.

Derive is, as usual, willing to tackle messy problems, with many equations and terrible-looking numbers.

keystrokes: A (for Author), type [2x+3y+5z = 11, 11x–13y+17z = –1, x+y+z = 23], Enter, L (for soLve), Enter, X (for approXimate), Enter, arrowup, arrowup, M (for Manage), S (for Substitute), Enter, tap Del to clear, type 1168/39, Enter, tap Del to clear, type 92/13, Enter, tap Del to clear, type –547/39, Enter, S (for Simplify), Enter.

5: $[2x + 3y + 5z = 11,\ 11x - 13y + 17z = -1,\ x + y + z = 23]$

6: $\left[x = \frac{1168}{39},\ y = \frac{92}{13},\ z = -\frac{547}{39}\right]$

7: $[x = 29.9487,\ y = 7.87692,\ z = -14.0256]$

8: $\left[2\,\frac{1168}{39} + 3\,\frac{92}{13} + 5\left[-\frac{547}{39}\right] = 11,\ 11\,\frac{1168}{39} - 13\,\frac{92}{13} + 17\left[-\frac{547}{39}\right] = -1,\ \frac{1168}{39}\right.$

9: $[11 = 11,\ -1 = -1,\ 23 = 23]$

You can also solve such equations, in general, for any coefficients.

keystrokes: A (for Author), type [ax+by = c, dx+ey = f], Enter, L (for soLve), Enter, Enter, Enter, arrowup, M (for Manage), S (for Substitute), Enter, tap Del to clear, type (ce–bf)/(ae–bd), Enter, tap Del to clear, type (af–cd)/(ae–bd), Enter, Enter, Enter, S (for Simplify), Enter.

10: $[a x + b y = c,\ d x + e y = f]$

11: $\left[x = \frac{c e - b f}{a e - b d},\ y = \frac{a f - c d}{a e - b d}\right]$

12: $\left[a\,\frac{c e - b f}{a e - b d} + b\,\frac{a f - c d}{a e - b d} = c,\ d\,\frac{c e - b f}{a e - b d} + e\,\frac{a f - c d}{a e - b d} = f\right]$

13: $[c = c,\ f = f]$

Try the same process with three equations and three unknowns but with general coefficients.

Chapter 7

Many Ways to Solve a Quadratic Equation

One way to see the richness of math and the varied strengths of Derive is by solving one problem many ways. A harmless looking problem like solve for x in the equation $x^2-5x+6=0$ will serve us nicely for two purposes. Before we start, think of how many ways you know to solve this equation.

Method 1: If we solve $x^2-5x+6=0$ for x^2 and then divide both sides of the equation by x, we get $x=5-6/x$. Substitute for the x on the right side of this equation the value of x on the left side of this equation. We get $5-6/(5-6/x)$. As you can imagine this process can repeated as far as we like. Graph $y=5-6/x$ (a hyperbola with 2 branches) and then graph $y=5-6/(5-6/x)$ (also a hyperbola with 2 branches). Where these graphs cross are solutions of the original equation, $x^2-5x+6=0$.

keystrokes: A (for Author), type x^2–5x+6=0, Enter, A (for Author), type 5x–6, Enter, B (for Build), Enter, type +, arrowup (to highlight x^2–5x+6=0), Enter, D (for Done), S (for Simplify), Enter.

```
1:   x^2 - 5 x + 6 = 0
2:   5 x - 6
3:   5 x - 6 + (x^2 - 5 x + 6 = 0)
4:   x^2 = 5 x - 6
```

Next we need to divide both sides of this equation by x and simplify.

keystrokes: A (for Author), type x, Enter, arrowup (to highlight x^2=5x–6), B (for Build), Enter, type /, arrowdown (to highlight x), Enter, D (for Done), S (for Simplify), Enter, E (for Expand), Enter.

5: x

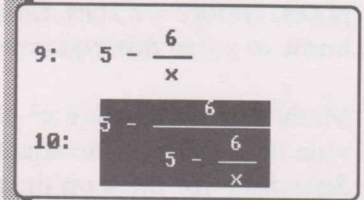
6: $\dfrac{x^2 = 5x - 6}{x}$

7: $x = \dfrac{5x-6}{x}$

8: $x = 5 - \dfrac{6}{x}$

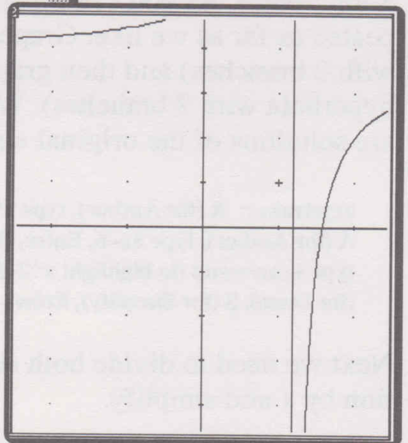
Now that we have our equation for x we will isolate the right side of the equation, make our substitution and plot.

> **keystrokes: Arrowright twice (to highlight 5–6/x), A (for Author), tap key F3 (to copy), Enter, M (for Manage), S (for Substitute), Enter, tap Del key (to clear), tap key F3 (to copy), Enter.**



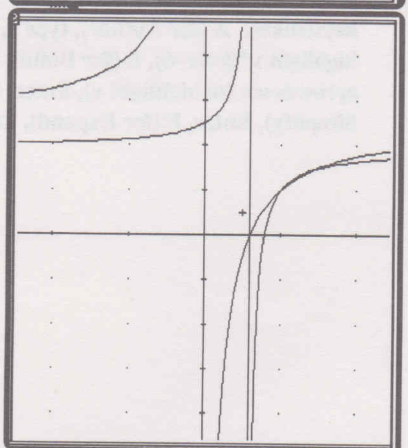
Now we need to graph these 2 expressions and see where they cross.

> **keystrokes: W (for Window), S (for Split), V (for Vertical), Enter, tap key F1 (to move to the next window), W (for Window), D (for Designate), 2 (for 2D–plot), Y (for Yes), P (for Plot).**



I can't see enough of this graph so I'll suggest we back up a bit and then put on the second graph. (for version 1.4 and later, the vertical line in these two graphs no longer appears . . . good!)

> **keystrokes: Tap key F10 (to back away), A (for Algebra), arrowup (to highlight 5–6/x), P (for Plot), P (for Plot).**

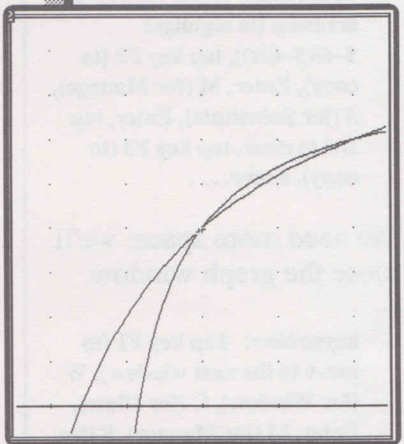


Now we'll use the very effective aim and zoom capabilities of Derive to look more closely at the points of the graphs we're most interested in.

By tapping the arrow keys we can move the + sign around the graphing window. As the + moves, look at the lower left part of your screen and you'll see the x and y values of the marker adjust as we go. Move your + marker, with the arrow keys, to where

the graphs appear to cross. We'll make that part of the graph the center of our window.

keystrokes: Arrowright and arrowup (to where the graphs cross), C (for Center), tap key F9 (to zoom in), use the arrow keys (to adjust the marker to a crossing point), tap F9 again, tap C (to Center).



When I move my marker with the arrow keys to the 2 crossing points I find x=2,y=2 at one point and x=3,y=3 at the other. So . . . x=2 and x=3 are the roots of the original equation.

Method 2. Here we work toward a solution by starting at some arbitrary number, compute our function's value at that number and recycle. The process is known as recursion. It's a wonderful process since we start at a wrong answer and march toward better and better answers. We'll start at x=5−6/x. Now we pick any number, like 11, substitute that for x on the right side and solve for x on the left side (these x's really become old x and new x). Take the result for x on the left and start over by substituting it on the right side. Amazingly enough, a pattern of numbers develops which heads off towards a solution.

9: $5 - \frac{6}{x}$

10: $5 - \frac{6}{5 - \frac{6}{x}}$

11: $5 - \frac{6}{x}$

12: $5 - \frac{6}{11}$

13: 4.45454

14: $5 - \frac{6}{4.45454}$

15: 3.65306

keystrokes: A (for Algebra), A (for Author), tap key F3 (to copy 5–6/x), Enter, M (for Manage), S (for Substitute), Enter, tap Del to clear, type 11, Enter, X (for approXimate), Enter, arrowup (to highlight 5–6/x), M (for Manage), S (for Substitute), Enter, tap Del to clear, type 4.45454, Enter, X (for approXimate), Enter.

If you continue this process you will generate a sequence of numbers which lead toward a solution. Try it! Start with a number different from 11 and do the recursive dance toward success.

Method 3. By creating a continued fraction

keystrokes: A (for Author), arrowup (to highlight 5–6/(5–6/x)), tap key F3 (to copy), Enter, M (for Manage), S (for Substitute), Enter, tap Del to clear, tap key F3 (to copy), Enter, . . .

16: $$5-\cfrac{6}{5-\cfrac{6}{x}}$$

17: $$5-\cfrac{6}{5-\cfrac{6}{5-\cfrac{6}{5-\cfrac{6}{x}}}}$$

We need more space: we'll close the graph window.

keystrokes: Tap key F1 (to move to the next window), W (for Window), C (for Close), Enter, M (for Manage), S (for Substitute), Enter, tap Del to clear, tap key F3 (to copy), Enter, M (for Manage), S (for Substitute), Enter, tap Del to clear, type 35, Enter, X (for approXimate), Enter.

18: $$5-\cfrac{6}{5-\cfrac{6}{5-\cfrac{6}{5-\cfrac{6}{5-\cfrac{6}{5-\cfrac{6}{5-\cfrac{6}{5-\cfrac{6}{x}}}}}}}}$$

19: $$5-\cfrac{6}{5-\cfrac{6}{5-\cfrac{6}{5-\cfrac{6}{5-\cfrac{6}{5-\cfrac{6}{5-\cfrac{6}{5-\cfrac{6}{35}}}}}}}}$$

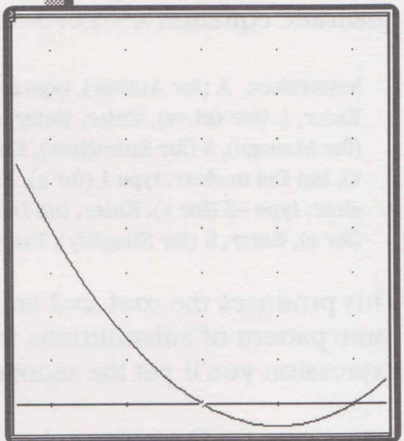
20: 3.03932

So, with x=35, the expression computes to 3.03932. Where does the 35 come from? I just made it up! A second number, 211, computes to 3.04040. I'll make a bet with you. You pick a number for x and compute the result in this big ugly expression and the result will be close to 3. Try it!

And 3 is a root of the original equation. Wow!

The following 4 methods are the more usual procedures for solving a quadratic equation. Derive is capable of doing the usual as well as the unusual.

Method 4. Graph $y=x^2-5x+6$. The values of x where the graph crosses the x axis are the numbers which solve the equation.



keystrokes: W (for Window), S (for Split), V (for Vertical), Enter, tap key F1 (to move to the next window), W (for Window), D (for Designate), 2 (for 2D–plot), y (for yes), A (for Algebra), A (for Author), type x^2–5x+6, Enter, P (for Plot), P (for Plot), tap key F10 if you can't see enough of the graph.

Method 5. Solve $x^2-5x+6=0$ directly.

keystrokes: A (for Algebra), A (for Author), tap key F3 (to copy x^2–5x+6), type =0, Enter, L (for soLve), Enter.

```
         2
21:  x    - 5 x + 6 = 0
22:  x = 2
23:  x = 3
```

Method 6. Factor $x^2-5x+6=0$ and set each factor to 0 and solve.

keystrokes: A (for Author), type x^2–5x+6, Enter, F (for Factor), Enter, R (for Rational), arrowleft (to highlight x–3), A (for Author), tap key F3 (to copy), type =0, Enter, L (for soLve), Enter, arrowup (to highlight (x–3)(x–2)), arrowright twice (to highlight x–2), A (for Author), tap key F3 (to copy), type =0, Enter, L (for soLve), Enter.

```
         2
24:  x    - 5 x + 6
25:  (x - 3) (x - 2)
26:  x - 3 = 0
27:  x = 3
28:  x - 2 = 0
29:  x = 2
```

Method 7. Using the quadratic formula to solve the

quadratic equation $x^2 - 5x + 6 = 0$.

> **keystrokes: A (for Author), type ax^2+bx+c=0, Enter, L (for soLve), Enter, Enter, arrowup once, M (for Manage), S (for Substitute), Enter, Enter (to skip x), tap Del to clear, type 1 (for a), Enter, tap Del to clear, type –5 (for v), Enter, tap Del to clear, type 6 (for c), Enter, S (for Simplify), Enter.**

1: $a x^2 + b x + c = 0$

2: $x = -\dfrac{\sqrt{(b^2 - 4 a c)} + b}{2 a}$

3: $x = \dfrac{\sqrt{(b^2 - 4 a c)} - b}{2 a}$

4: $x = -\dfrac{\sqrt{((-5)^2 - 4\ 1\ 6)} + -5}{2\ 1}$

5: $x = 2$

This produces the root x=2 and if you do the same pattern of substitutions in the other expression you'll get the second solution.

A last way for Derive to solve $x^2 - 5x + 6 = 0$ is a more automated method similar to method 1. The command iterates makes a list of numbers or expressions which result from substituting a value (or expression) for a variable then calculating a result and feeding that result back in.

> **keystrokes: A (for Author), type iterates (5–6/x, x, 1.8, 6), Enter, X (for approX), Enter.**

1: $\text{ITERATES}\left[5 - \dfrac{6}{x}, x, 1.8, 9\right]$

2: [1.8, 1.66666, 1.4, 0.714285, -3.4, 6.76470, 4.11304, 3.54122, 3.30567, 3.18493]

Gradually make the last number bigger. And then try approX with no last term. Also try the command iterate, which will produce the nth term and not a list.

Chapter 8

Complex Numbers

A look at history has often helped me understand the math I was studying. Nowhere is that more true than with complex numbers. First of all, the name is a problem . . . complex numbers. It sounds very difficult–something only Gauss or Riemann or von Neumann could understand.

Well, let's see if that's true. In the past, people could solve an equation like $3+x = 12$ by choosing a 9 for the x, thus making the true statement $3+9 = 12$. However, $x+5 = 2$ had no solution for a long time. Only when negative numbers were invented and accepted could a solution of -3 exist. $-3+5 = 2$ makes a true statement. It's hard for me to believe but until the seventeenth century negative numbers were not fully incorporated into the real number system. There is a direct analogy between this historical experience and the invention and acceptance of complex numbers.

If we solve $x^2 = 25$ we notice two solutions, 5 or -5. We can see this by inspection or by seeing where the graph of this function crosses the x axis. Another choice would be to use soLve on Derive:

> **keystrokes: A (for Author), type x^2=25, Enter, L (for soLve), Enter.**

```
         2
1:      x  = 25
2:      x = 5
3:      x = -5
```

Try solving $x^2 = 4$ or $x^2 = 1$ by inspection and by Derive. In both examples solutions are easy. But now try $x^2 = -1$. By inspection it would seem 1 or -1 are the only possibilities. But $1^2 = 1$ and $(-1)^2 = 1$. So there seems to be no solution. Now let Derive try.

> **keystrokes: A (for Author), type x^2=–1, Enter, L (for soLve), Enter.**

```
         2
4:      x  = -1
5:      x = î
6:      x = - î
```

Derive gives two solutions, neither of which looks like what was called a number for most of our history. Let's test these two solutions.

keystrokes: Arrowup twice (to highlight x^2 = –1), M (for Manage), S (for Substitute), Enter, arrowdown (to highlight x = î), arrowright once or twice (to highlight î), tap Del key to clear, tap key F3(to copy the highlighted expression), Enter, S (for Simplify), Enter.

```
7:    î² = -1
8:    -1 = -1
```

So î seems to work . . . let's try –î.

keystrokes: Arrowup 4 times (to highlight x^2 = –1), M (for Manage), S (for Substitute), Enter, arrowdown twice (to highlight x = –î), arrowright twice (to highlight – î), tap Del key to clear, tap key F3 (to copy the highlighted expression), Enter, S (for Simplify), Enter.

```
9:    (- î)² = -1
10:   -1 = -1
```

Here are two solutions for a problem that had no solution for thousands of years. By 1800 this idea was just appearing and by 1900 it was widely accepted.

Unfortunately, these new numbers were called imaginary numbers; this gave them an unnecessary air of mystery. If we look back in time, the invention of 3 or –5 or 2/3 or 0 is even more amazing than the invention of imaginary numbers.

In some sense all "numbers" are "imaginary"; that is, they are things we make up in our minds and connect to parts of the world.

People who are practical sometimes question the math person's contact with reality. If we're counting Porsches, then numbers like 1, 2, 3 do it quite well: we don't often talk about 2 and 1/2 Porsches or –2 Ferrari's. Numbers like 1, 2, 3 can do these jobs quite well.

But if I'm buying a movie ticket I may need 4 and 1/2 dollars or if I owe you money we could talk about −$10 (or multiplying by some billions, the present U.S. annual trade deficit).

Imaginary numbers like î or 5î and complex numbers like 3+2î all have reasonable physical representations just as other numbers do. They are used every day by practical people like electronic engineers and physicists (but, so far, not by even the most creative accountants).

Let's explore these new entities by examining powers of î.

keystrokes: A (for Author), hold down the Alt key and type i, Enter.

keystrokes: A (for Author), hold down the Alt key and type i, hold shift and type 6 (for ^), type 2, Enter, S (for Simplify), Enter.

```
11:  î

      2
12:  î

13:  -1
```

So $\hat{\imath}^{1}$ is î and $\hat{\imath}^{2} = -1$. Now try $\hat{\imath}^{3}$ and $\hat{\imath}^{4}$. Simplified, can you guess the result? Keep going and look for patterns. What is $\hat{\imath}^{10}$? Predict from your experience. What is $\hat{\imath}^{17}$?

Try $\hat{\imath}^{4}$, $\hat{\imath}^{8}$, $\hat{\imath}^{12}$, $\hat{\imath}^{16}$ . . .

Try $\hat{\imath}^{2}$, $\hat{\imath}^{6}$, $\hat{\imath}^{10}$ . . .

. . . what is $\hat{\imath}^{100}$? $\hat{\imath}^{101}$? $\hat{\imath}^{99}$?

And what is $\hat{\imath}^{(-1)}$? $\hat{\imath}^{(-2)}$? $\hat{\imath}^{(-3)}$? $\hat{\imath}^{\hat{\imath}}$, $\hat{\imath}^{(2\hat{\imath})}$, $\hat{\imath}^{(3\hat{\imath})}$ . . .

Now that you are familiar with the natural number powers of î and the patterns they make, we can try the same idea with powers of the complex number 1+î.

> **keystrokes: A (for Author), type (1+, hold down Alt key and type i, type)^1, Enter, S (for Simplify), Enter.**

```
14:   (1 + î)^1
15:   1 + î
```

So $(1+\hat{\imath})^1$ is 1+î.

> **keystrokes: A (for Author), type (1+, hold down Alt key and type i, type)^2, Enter, S (for Simplify), Enter.**

```
16:   (1 + î)^2
17:   2 î
```

Here is another case where Derive is too quick and powerful for us to see what's happening. What rule for multiplying complex numbers is Derive using?

We'll try $(1+a)^2$ and then substitute î to see the details.

> **keystrokes: A (for Author), type (1+a)^2, Enter, E (for Expand), Enter.**

```
18:   (1 + a)^2
19:   a^2 + 2 a + 1
```

So $(1+a)^2$ is a^2+2a+1 and we can substitute î for a.

> **keystrokes: M (for Manage), S (for Substitute), Enter, hold down Alt key and type i, Enter, arrowleft (to highlight $\hat{\imath}^2$), S (for Simplify), Enter.**

```
20:   î^2 + 2 î + 1
21:   -1 + 2 î + 1
22:   2 î
```

This makes sense since $\hat{\imath}^2$ is −1. Now to simplify the whole expression.

> **keystrokes: S (for Simplify), Enter.**

Let's try $(1+\hat{\imath})^3$.

> **keystrokes: A (for Author), type (1+, hold down Alt key and type i, type)^3, Enter, E (for Expand), Enter.**

```
23:   (1 + î)^3
24:   -2 + 2 î
```

And using our slow–motion trick . . .

keystrokes: A (for Author), type (1+a)^3, Enter, E (for Expand), Enter, M (for Manage), S (for Substitute), Enter, hold down the Alt key and type i, Enter, arrowright (to highlight i^3), S (for Simplify), Enter, S (for Simplify), Enter.

$$
\begin{array}{ll}
25: & (1 + a)^3 \\
26: & a^3 + 3 a^2 + 3 a + 1 \\
27: & \hat{\imath}^3 + 3 \hat{\imath}^2 + 3 \hat{\imath} + 1 \\
28: & - \hat{\imath} + 3 \hat{\imath}^2 + 3 \hat{\imath} + 1 \\
29: & -2 + 2 \hat{\imath}
\end{array}
$$

This slow–motion process can be helpful in many situations. Derive can help us to explore in enough detail that we can actually learn some math rather than just accept whatever appears. Derive can produce answers quickly or it can go slowly and show the process clearly. We are driving, so we need to choose the speed to suit our current goals.

The pattern for $(1+\hat{\imath})^n$ does not seem to be as easy to follow as $\hat{\imath}^n$. Recall that $\hat{\imath}^1$ . . . $\hat{\imath}^8$ went $\hat{\imath}, -1, -\hat{\imath}, 1, \hat{\imath}, -1, -\hat{\imath}, 1$. We should expect something like this with $(1+\hat{\imath})^n$ although not exactly the same. As a start we'll write $(1+\hat{\imath})^n$ with n from 1 to 4:

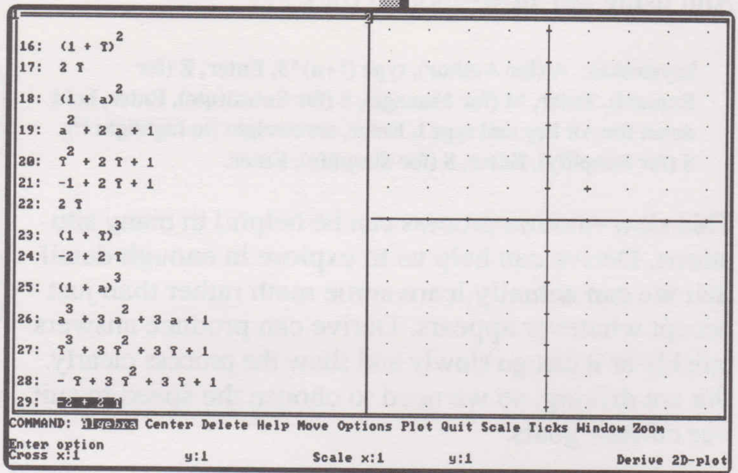
$(1+\hat{\imath})^1 = 1+\hat{\imath}$ $(1+\hat{\imath})^2 = 0+2\hat{\imath}$

$(1+\hat{\imath})^3 = -2+2\hat{\imath}$ $(1+\hat{\imath})^4 = -4+0\hat{\imath}$

This form is called rectangular or Cartesian form or standard form for complex numbers. There's an advantage . . . we have ordered pairs which allow us to think of normal two dimensional graphs. To guide our thinking, we'll graph $\hat{\imath}^n$, a known pattern, and then we'll graph $(1+\hat{\imath})^n$. First we'll set up a graph window.

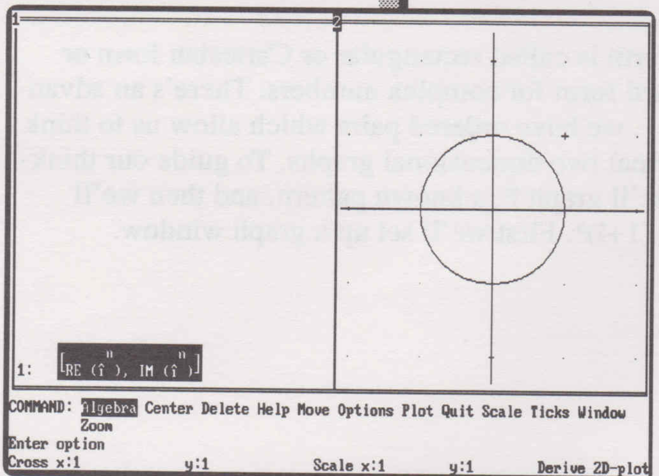
keystrokes: W (for Window), S (for Split), V (for Vertical), Enter, tap key F1 (to change windows), W (for Window), D (for Designate), 2 (for 2D–plot), y (for yes).

This gives us a graphing window. Now we can plot $\hat{\imath}^n$.



keystrokes: A (for Algebra), A (for Author), type [re(, hold down Alt key and type i, type ^n), im(, hold down Alt key and type i, type ^n)], Enter.

This allows us to pull out the so–called real part of $\hat{\imath}^n$ and plot it as the first number of an ordered pair of real numbers, and then pull out and plot the imaginary part (the number times $\hat{\imath}$) as the second number. The square brackets will allow us to plot the points. The whole expression is called parametric form.



keystrokes: P (for Plot), S (for Scale), type 1, Tab, type 1, Enter, P (for Plot), hold the Del to clear, type 0, Tab, hold down the Del key to clear, type 4, Enter.

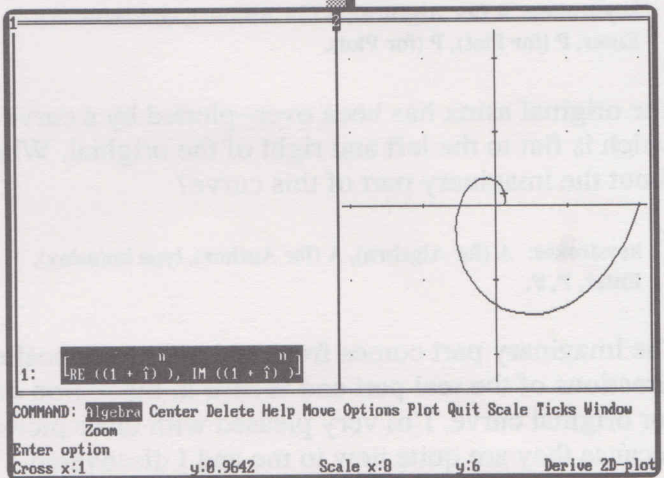
(By the way, you should get a nice smooth circle. If you get an ellipse or oval,

change your Ticks to 8 and 8 or some other numbers until the circle looks right . . . monitors are different and Tick settings allow us to make an adjustment.)

What really happens is that lots of points are generated by numbers for n between 0 and 4. If we plotted only n = 1, 2, 3, 4 we'd get four points which happen to lie on a circle. We'll use this important feature of Derive's graphing program in Chapter 9.

What about the graph of $(1+\hat{\imath})^n$?

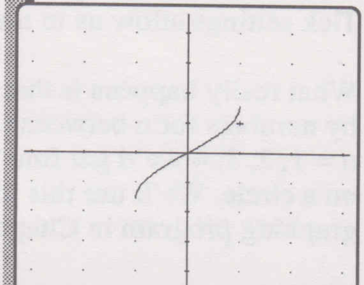
keystrokes: D (for Delete), A (for All), S (for Scale), tap Del to clear, type 8, Tab, tap Del to clear, type 6, Enter, A (for Algebra), A (for Author), type [re((1+, hold down Alt key and type i, type)^n), im((1+, hold down Alt key and type i, type)^n)], Enter, P (for Plot), P (for Plot), Tab, tap Del to clear, type 8, Enter.



So $\hat{\imath}^n$ is a nice smooth circle of radius 1 and center at (0,0) and $(1+\hat{\imath})^n$ is a gradually expanding nice smooth curve called a spiral. A circle can be thought of as a spiral whose radius does not change. A spiral can be thought of as a circle whose radius does change.

The command for the real part of a number is re and the corresponding command for the imaginary part is im. These can offer many possibilities in graphing.

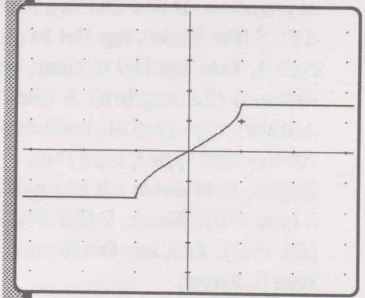
The arc sine function can be interesting. I read it as "the arc whose sine is x". Asin(1/2) is the arc whose sin is 1/2; π/6 or 30 degrees would be correct. Let's look at the graph of asin(x).



keystrokes: A (for Algebra), A (for Author), type asinx, Enter, P (for Plot), D (for Delete), A (for All), S (for Scale), hold Del to clear, type 1, Tab, hold Del to clear, type 1, P (for Plot).

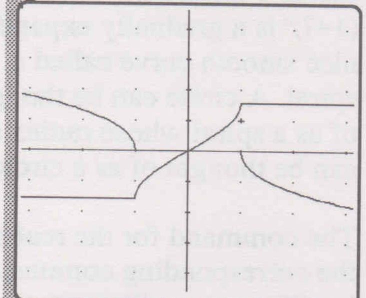
This graph always seemed strange to me . . . cut off. Where is the rest? Let's look around.

keystrokes: A (for Algebra), A (for Author), type re(asinx), Enter, P (for Plot), P (for Plot).



The original asinx has been over–plotted by a curve which is flat to the left and right of the original. What about the imaginary part of this curve?

keystrokes: A (for Algebra), A (for Author), type im(asinx), Enter, P, P.



The imaginary part comes from and goes to opposite directions of the real part and is zero in the region of our original curve. I'm very pleased with these pictures because they are quite new to me and I discovered them quite by chance while experimenting with combinations of commands in Derive.

What about the plot of re(sin(x+xî))? We'll be looking at the sine of complex numbers like −2−2î and −1−î and 3+3î and then only at the real part of the sine of those numbers. Try it! Superimpose the im(sin(x+xî)) and compare them. Did the graphs look like what you expected? Try im(sin(x+î)) and im(sin(x+2î)). Do vector(im(sin(x+k*î)), k, 4). Don't forget to Simplify before plotting.

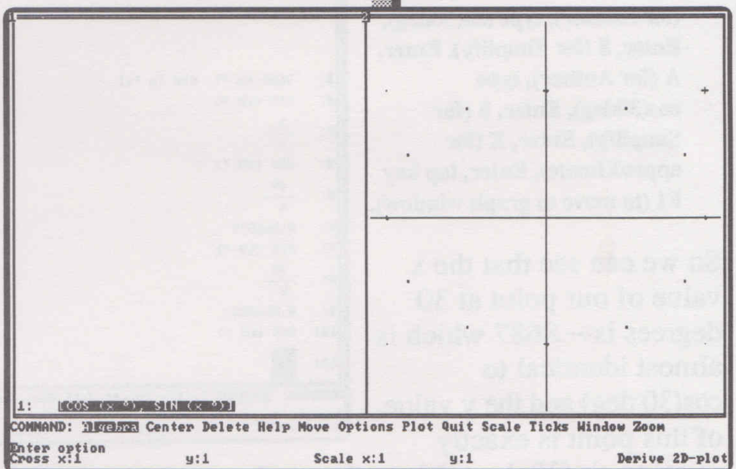
Chapter 9

Trigonometry

One of the clearest ways to see much of trigonometry is to focus on basic characteristics of a circle of radius 1, the so–called unit circle.

First we'll set up 12 points which are evenly spaced on such a circle. Since there are 360 degrees in a circle, the points are usually labeled 0 degrees at the extreme right and then 30 degrees for the first point counterclockwise and then 60 degrees and 90 degrees and on around to 360 degrees, which brings us back to the original point. This simple picture will allow us to develop a lot of trig.

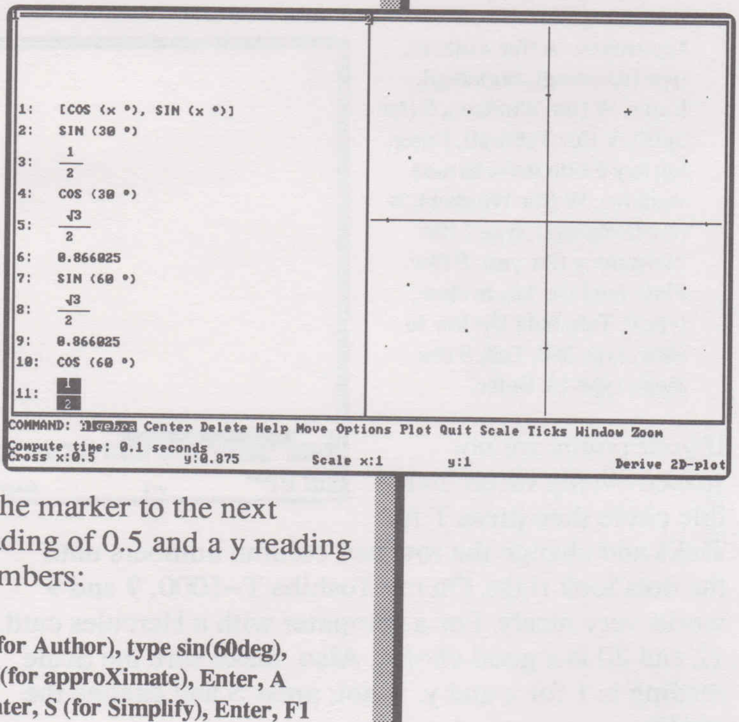
keystrokes: A (for Author), type [cos(xdeg), sin(xdeg)], Enter, W (for Window), S (for Split), V (for Vertical), Enter, tap key F1 (to move to next window), W (for Window), D (for Designate), type 2 (for 2D–plot), y (for yes), P (for Plot), hold Del key to clear, type 0, Tab, hold Del key to clear, type 360, Tab, S (for Step), type 13, Enter.



If your points are not spaced evenly on an invisible circle then press T for Ticks and change the row and column numbers until the dots look right. On my Toshiba T–1000, 9 and 9 works very nicely. For a computer with a Hercules card, 12 and 20 is a good choice. Also, make sure the Scale reading is 1 for x and y. If not, press S and change the reading.

Now move the marker (+) on the graph window by pressing the arrow keys. (If the marker doesn't move, press key F1 to move into the graph window). Move the marker (+) to the first point up from the middle on the right side (30 degrees). When I do this, my x reading (lower left of the screen) is .8687 and my y is .5. Your reading can vary by a small amount depending on your monitor and/or how close you place the marker to the 30 degree point on the circle. Let's compare these values with the sin(30 deg) and cos(30 deg).

keystrokes: A (for Algebra), A (for Author), type sin(30deg), Enter, S (for Simplify), Enter, A (for Author), type cos(30deg), Enter, S (for Simplify), Enter, X (for approXimate), Enter, tap key F1 (to move to graph window).



So we can see that the x value of our point at 30 degrees is −.8687 which is almost identical to cos(30 deg) and the y value of this point is exactly equal to sin(30 deg). Move the marker to the next point (60 deg). I get an x reading of 0.5 and a y reading of 0.875. We'll check the numbers:

keystrokes: A (for Algebra), A (for Author), type sin(60deg), Enter, S (for Simplify), Enter, X (for approXimate), Enter, A (for Author), type cos(60deg), Enter, S (for Simplify), Enter, F1 (to move to the graph window).

We see that sin(60deg) is almost exactly the same as the y value at 60 degrees and cos(60 deg) is exactly the same as the x value at 60 degrees. The x and y values

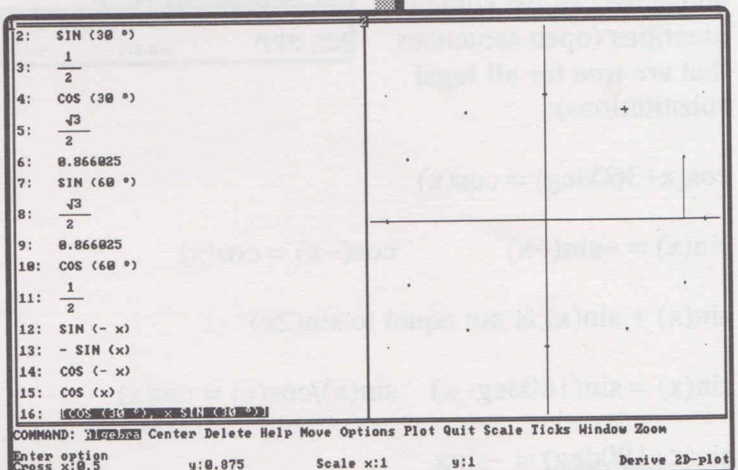
represent the perpendicular distances from the axes to that 60 degree point on our unit circle.

Combining these two ideas, we see that sine and cosine of an arc are just fancy names for the x and y coordinates of the point on a unit circle (centered at (0,0)) at that arc distance from the 0 point on the circle.

Since the circle on the screen has a radius of 1, the sin(90 deg) must be 1; try it! What is the sin(270 deg)? cos(180 deg)? Did you notice the sin(30 deg) was the same as the cos(60 deg)? Can you see that result from the points on the circle? Compare sin(60 deg) and sin(120 deg). Compare sin(30 deg) and sin(–30 deg). Also, cos(30 deg) and cos(–30 deg). How would sin(x) and sin(–x) compare? Look at the points on the circle to try to answer these questions and also use the algebra window.

keystrokes: A (for Algebra), A (for Author), type sin(–x), Enter, S (for Simplify), Enter.

keystrokes: A (for Author), type cos(–x), Enter, S (for Simplify), Enter.



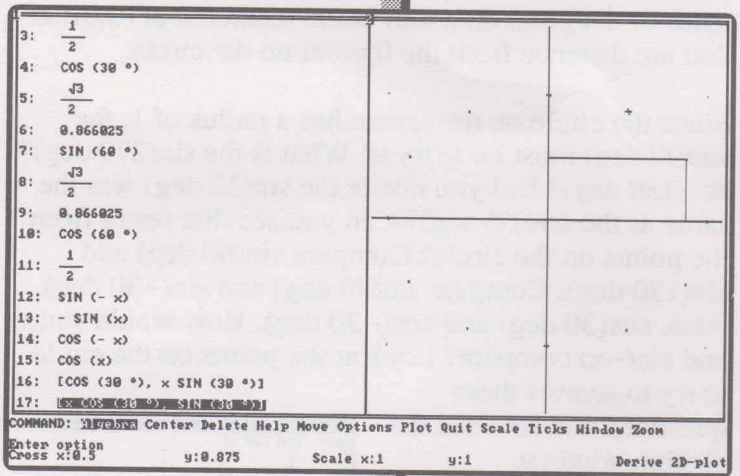
It is possible to have Derive show the segments which represent sine and cosine on the points of the circle:

keystrokes: A (for Author), type all of the following inside the square brackets including the comma: [cos(30deg), x sin(30deg)], Enter, P (for Plot), P (for Plot), hold Del key until clear, type 0, Tab, hold Del until clear, type 1, Tab, C (for Continuous), Enter.

This segment represents sin(30 deg). Try for sin(150 deg) and sin(60 deg) and sin(−60 deg) and . . .

To represent cos(30 deg):

> **keystrokes: A (for Algebra), A (for Author), type [x cos(30deg), sin(30deg)], Enter, P (for Plot), P (for Plot), Enter.**



Try for cos(−30 degrees),
cos(150 degrees),
cos(300 degrees),
cos(780 degrees).

Based on our experience above we can list some trig identities (open sentences that are true for all legal substitutions):

cos(x+360deg) = cos(x)

sin(x) = −sin(−x)   cos(−x) = cos(x)

sin(x) + sin(x) is not equal to sin(2x)

sin(x) = sin(180deg−x)   sin(x)/cos(x) = tan(x)

sin(x+180deg) = −sinx

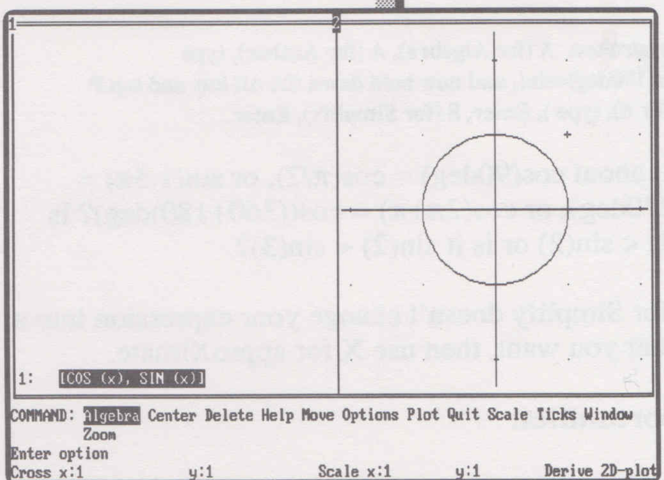
sec(x) = 1/cos(x)

Make up lots more of these yourself. Look at the unit circle for clues. Make many errors and you will learn.

So far, we have measured our locations on the unit circle by degrees, with 360 degrees to each complete turn. A second way of measuring locations on a unit circle is by measuring the actual distance from a fixed reference point on the circle (usually at the far right of the circle). Since the distance around any circle is 2π radius, and since, by definition, the radius of a unit circle is 1, then the distance around a unit circle is 2π. Since π is about 3.14, then the distance around is about 6.28. If we measure locations on a unit circle by distance, the unit is called radians. 1 radian is about 1/6 of the way around the circle or about 60 degrees: 2π radians takes us all the way around, so it must be the same as 360 degrees.

The following keystrokes will construct a complete unit circle using radian measure instead of degrees:

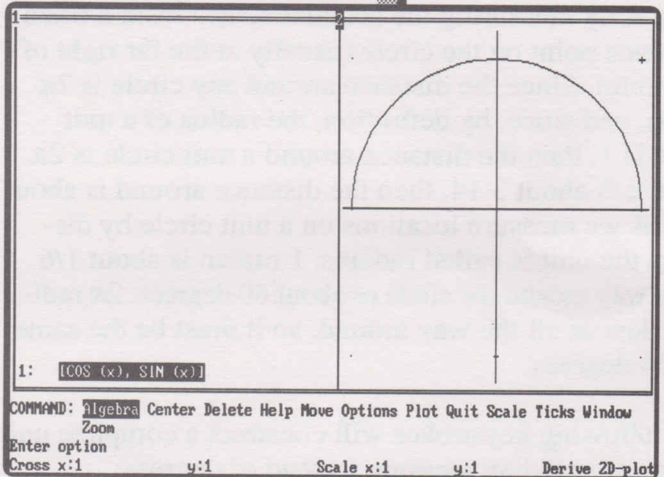
keystrokes: Tap F1 to move to the graph window (if necessary), D (for Delete), A (for All), A (for Algebra), A (for Author), type [cosx,sinx], Enter, P, P, hold down Del key until clear, type 0, Tab, hold down Del key until clear, type 2, hold down the Alt key and tap P (for π), Tab, C (for Continuous), Enter.



So, if we use sin(x) or cos(2x) with no deg in the parentheses, then we are automatically using radians, and 0 to 2π gives a full circle. If we want to use degrees as our unit then we use sin(xdeg) or cos(2xdeg) and 0 to 360 is a full circle.

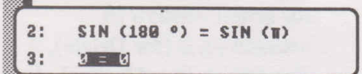
The following suggestions may help you to see how to handle radians on Derive.

keystrokes: F1 to the graph window if you're not there already, D (for Delete), A (for All), P (for Plot), Tab, tap Del key once to remove the 2, Enter.



By going from 0 to pi we constructed 1/2 a circle, just as when we went from 0 degrees to 180 degrees. So, sin(180deg) should be the same as sin(π).

keystrokes: A (for Algebra), A (for Author), type sin(180deg)=sin(, and now hold down the Alt key and tap P (for π), type), Enter, S (for Simplify), Enter.



What about cos(90deg) = cos(π/2), or sin(1.5π) = sin(270deg), or cos(2π+π) = cos((360+180)deg)? Is sin(3) < sin(2) or is it sin(2) < sin(3)?

If S for Simplify doesn't change your expression into a number you want, then use X for approXimate.

Opportunities:

Does sin((30deg)+(60deg)) = sin(30deg)+sin(60deg)? Look at the unit circle to understand what is happening. Make up more examples in which a statement you might expect to be true is in fact false. Clear your graph screen and plot [cos(x), sin(x)] from 0 to π/6; from 0 to

2 $\pi/6$; from 0 to 3 $\pi/6$ . . . Try to get the same effect using [cos(xdeg), sin(xdeg)]. Use Delete All before starting a new graph.

Does sin(2deg) = cos(2deg)? Look at the unit circle to see why they're not equal. Can you find an x so that sin(2deg) = cos(xdeg). Can you find more than one value for x? For what x does sin(3deg) = cos(xdeg)? . . . does sin(4deg) = cos(xdeg)? Can you generalize this experience so that for any k you can predict an x in sin(kdeg) = cos(xdeg)?

Which of the following statements are true?
If a statement is false then fix it up so it is true.
If true, then make up more examples like mine.

sin(91deg) = sin(89deg)

cos(179deg) = cos(1deg)

sin($\pi/6$) = sin(13 $\pi/6$) = sin(25 $\pi/6$) = sin(37 $\pi/6$)

tan(2$\pi/6$) = tan(60deg)

tan(45deg) = sin(45deg)/cos(45deg)

30deg = $\pi/6$ = 13 $\pi/6$

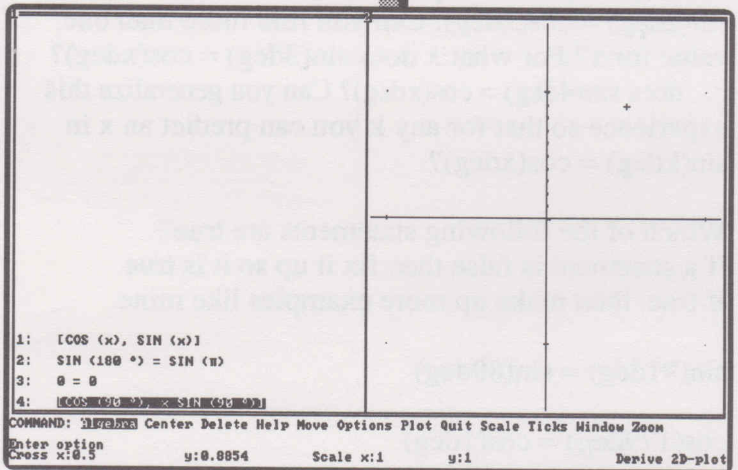
sin(30deg) = sin(13 $\pi/6$)

sin(10deg) =? sin(20deg) =? sin(30deg) =?

Continue this sequence and look for patterns.
Do the same for cos(x), tan(x) (which is sin(x)/cos(x)), cot(x) (which is 1/tan(x)), sec(x) (which is defined as 1/cos(x)), and csc(x) (which is 1/sin(x)).

Clear your graph screen with Delete All (remember to be in the graph screen before you take action-look for the shaded number at the top of the window and use F1 to shift between windows) and experiment with representations of sin(x) and cos(x).

> **keystrokes: A (for Algebra), A (for Author), type [cos(90deg), xsin(90deg)], Enter, P, P, hold down Del key until clear, type 0, Tab, hold down Del key until clear, type 1, Tab, S (for Step), type 11, Enter.**

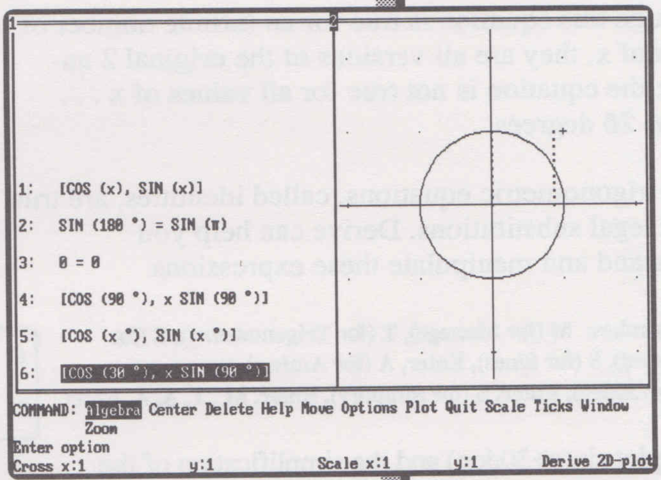


If you'll look closely at your vertical or y axis you'll find a nice "ruler", separated into tenths. Since sin(90deg) is 1, each of the marks on the "ruler" represents 1/10. Now plot the following unit circle.

> **keystrokes: A, A, type [cos(xdeg),sin(xdeg)], Enter, P, P, hold Del to clear, type 0, Tab, hold Del to clear, type 360, Tab, C (for Continuous), Enter.**

Now we can put our "ruler" where we want to on the unit circle to measure the value of the sine.

keystrokes: A (for Algebra), arrowup to [cos(90deg), xsin(90deg)], A (for Author), press F3 key (to copy), hold Ctrl key down and press key A until the underline moves to the 9 of cos(90deg) and then type 3, Enter, P, P, hold Del to clear, type 0 , Tab, hold Del to clear, type 1, Tab, S (for Step), hold Del to clear, type 11, Enter.



Now, if you count spaces from the bottom, remembering that each space is 1/10, you'll get about 5/10 as the distance from the horizontal midline of the circle to the crossing of the circle. So, we confirm that the sin(30deg) is 1/2. Try moving the ruler to 60deg or 120deg or . . . Try making a horizontal "ruler" for exploring cos. (You could make life easier if you made a ruler from paper and used that on your screen).

Trigonometric Identities

An infinite number of algebraic equations are true for only certain values of their variable; for examples $2x+5=13$ is true for only $x=4$ and $x^2-1=24$ is true only for $x=5$ or -5.

Another infinite number of algebraic equations are true for all legal substitutions: $x+x=2x$ and $(x+3)^2=x^2+6x+9$ are always true.There are analogies in trigonometric equations.

cos(xdeg)=1/2 is true for x=60 degrees or x=300 degrees or x=420 degrees or x=660 degrees and so on.

Although this equation is true for an infinite number of values of x, they are all versions of the original 2 answers; the equation is not true for all values of x . . . such as 26 degrees.

Other trigonometric equations, called identities, are true for all legal substitutions. Derive can help you understand and manipulate these expressions.

keystrokes: M (for Manage), T (for Trigonometry), E (for Expand), S (for Sines), Enter, A (for Author), type sin(x+30deg), Enter, S (for Simplify), Enter, M , T, A, A, Enter.

1: $\mathrm{SIN}\ (x + 30\ °)$

2: $\dfrac{\mathrm{COS}\ (x)}{2} + \dfrac{\sqrt{3}\ \mathrm{SIN}\ (x)}{2}$

If you plot sin(x+30deg) and the simplification of the same you'll see that the graphs look identical. If you substitute random numbers for x in each expression and approXimate, you'll get the same results.

With the help of the vector command we can make a list of expressions; with the help of square brackets we can display this list in a table.

keystrokes: A (for Author), type vector([sin(n*x)], n, 4), Enter, S (for Simplify), Enter, M (for Manage), T (for Trigonometry), E (for Expand), S (for Sines), Enter, S (for Simplify), Enter.

3: VECTOR ([SIN (n x)], n, 4)

4: $\begin{bmatrix} \mathrm{SIN}\ (x) \\ \mathrm{SIN}\ (2\ x) \\ \mathrm{SIN}\ (3\ x) \\ \mathrm{SIN}\ (4\ x) \end{bmatrix}$

5: $\begin{bmatrix} \mathrm{SIN}\ (x) \\ 2\ \mathrm{SIN}\ (x)\ \mathrm{COS}\ (x) \\ 3\ \mathrm{SIN}\ (x) - 4\ \mathrm{SIN}\ (x)^3 \\ (4\ \mathrm{SIN}\ (x) - 8\ \mathrm{SIN}\ (x)^3)\ \mathrm{COS}\ (x) \end{bmatrix}$

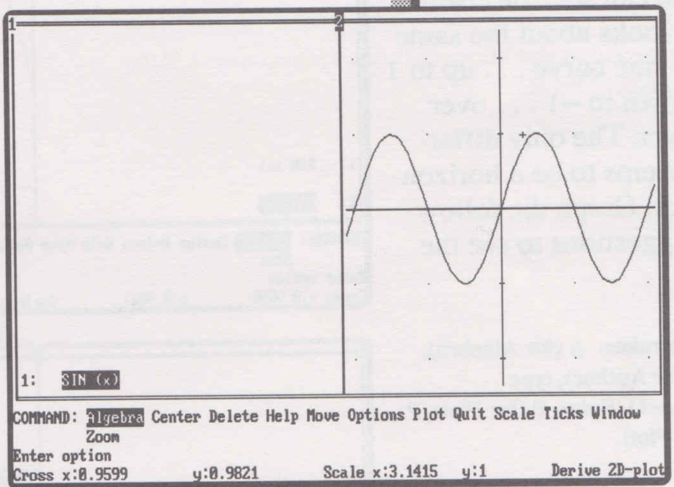
Return the settings to Auto, which is the default case, when you have finished experimenting. We've only begun to explore the possibilities here.

Chapter 10

Graphing Trig Functions

Like most graphs, the graph of sin(x) = y goes on and on to the left and the right. We are flying over an electronic field of graphs when we look at a graph on Derive. We are seeing only a small part of the graph. By pressing key F9 or F10 we move closer to the graph or back away to see a wider part of the graph. First we'll set up our graphing window and do a trig graph.

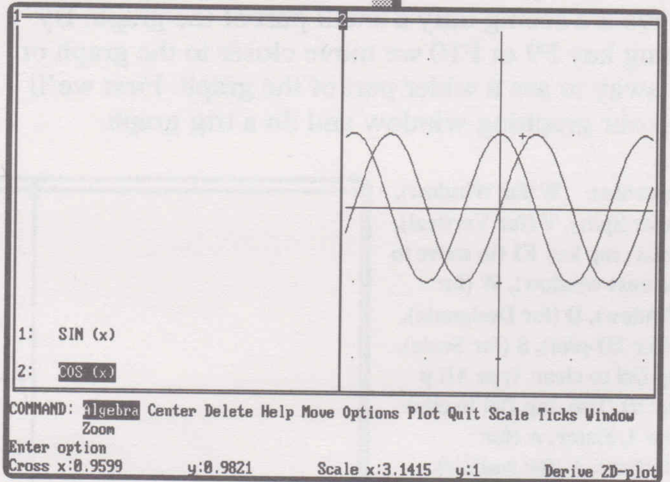
keystrokes: W (for Window), S (for Split), V (for Vertical), Enter, tap key F1 (to move to the next window), W (for Window), D (for Designate), 2 (for 2D-plot), S (for Scale), tap Del to clear, type Alt p (for π), Tab, tap Del to clear, type 1, Enter, A (for Algebra), A (for Author), type sinx, Enter, P (for Plot), P (for Plot).



Each mark on your grid horizontally is worth π and each mark vertically is worth 1. We see that the highest the sine curve reaches is 1 and the lowest is -1. We also see that the curve repeats approximately every 2 screen units horizontally, since each screen unit is worth π, then repeating begins every 2π. You can use the marker (+) on your screen and the corresponding x and y numbers at the lower left part of your screen to find the coordinates that match points on the screen. Try it! Use PgUp and PgDn and hold Ctrl key down and arrow left or right for bigger jumps. Use arrow keys only if you want small jumps.

Now we'll put a cosine curve on top of the sine curve and try to make some comparisons.

keystrokes: A (for Algebra), A (for Author), type cosx, Enter, P (for Plot), P (for Plot).

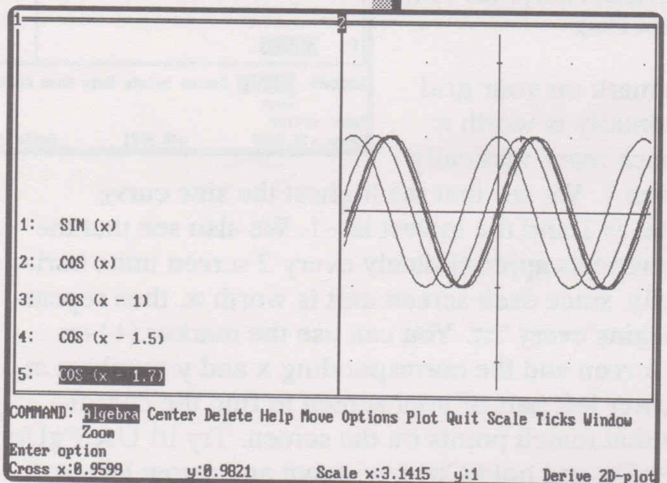


As you can see, the cosine curve looks about the same as the sine curve . . . up to 1 and down to −1 . . . over and over. The only difference seems to be a horizontal shift. Graph the following suggestions to see the trend.

keystrokes: A (for Algebra), A (for Author), type cos(x–1), Enter, P (for Plot), P (for Plot).

keystrokes: A, A, type cos(x–1.5), Enter, P, P.

keystrokes: A, A, type cos(x–1.7), Enter, P, P.



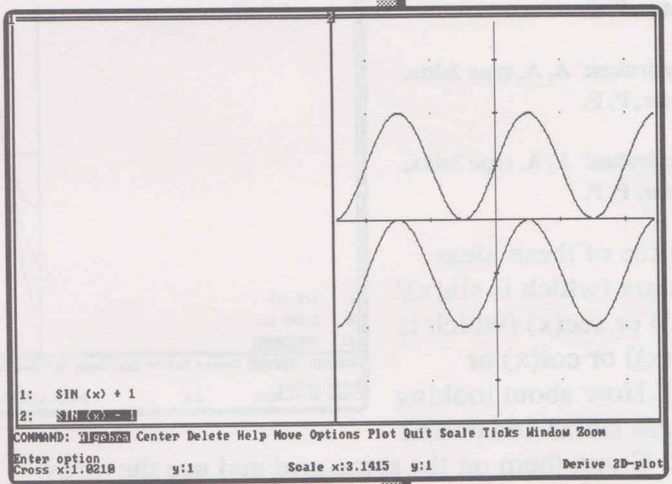
What value of the variable k, in cos(x–k), shifts the cosine curve on top of the sine curve? Can you find more than one value? Can you shift sine to cosine? Try . . . try . . . guess a lot, don't be afraid to make "mistakes". The best students are very experimental. Weaker students are too cautious.

Opportunities:

keystrokes: D (for Delete), A (for All), A, A, type sinx+1, Enter, P, P.

keystrokes: A, A, type sinx–1, Enter, P, P.

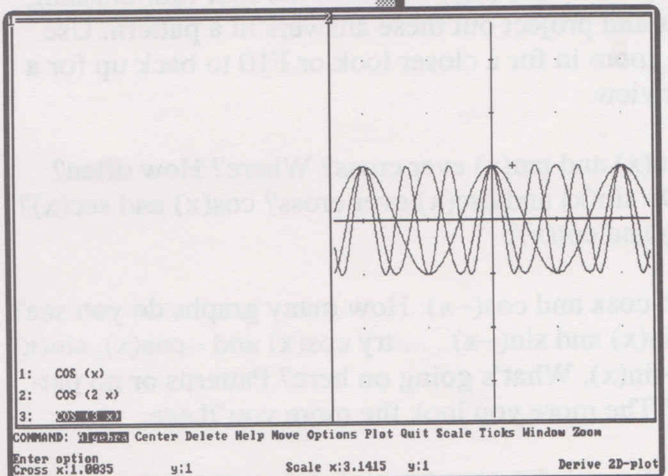
Patterns? . . . Variations? . . . You try!



keystrokes: D (for Delete), A (for All), A, A, type cosx, Enter, P, P.

keystrokes: A, A, type cos(2x), Enter, P, P.

keystrokes: A, A, type cos(3x), Enter, P, P.

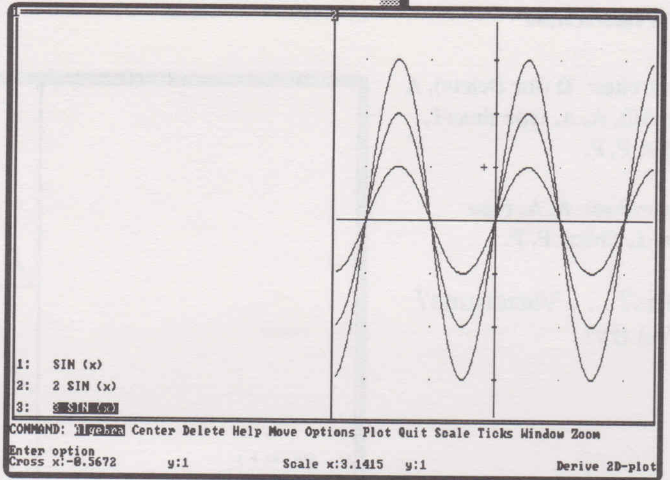


See anything? . . . patterns? . . . ideas? . . . experiment! You can always delete some or all and redo for a clearer look.

keystrokes: D (for Delete), A (for All), A, A, type sinx, Enter, P, P.

keystrokes: A, A, type 2sinx, Enter, P, P.

keystrokes: A, A, type 3sinx, Enter, P, P.



Try some of these ideas with tanx (which is sin(x)/cos(x)) or sec(x) (which is 1/cos(x)) or cot(x) or csc(x). How about looking at where sin(x) and cos(x) cross? Graph them on the same grid and use the arrow keys to locate the marker (+) at the crossing points. Make a list of the (x,y) values at the first four crossing points and project out these answers in a pattern. Use F9 to zoom in for a closer look or F10 to back up for a wider view.

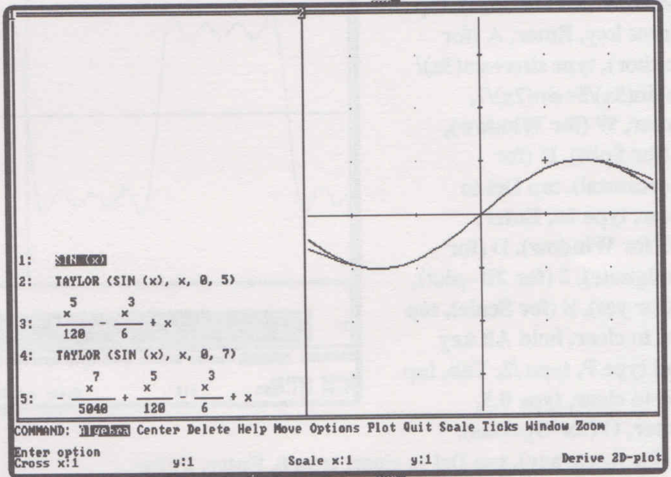
Do sin(x) and tan(x) ever cross? Where? How often? etc. Do sin(x) and csc(x) ever cross? cos(x) and sec(x)? tan(x) and cot(x)?

Graph cosx and cos(−x). How many graphs do you see? Try sin(x) and sin(−x) . . . try cos(x) and −cos(x), sin(x) and −sin(x). What's going on here? Patterns or no patterns? The more you look the more you'll see.

What about sin(x) + cos(x) or sin(x) − cos(x) or . . . experiment a lot. Where do graphs cross? When do they repeat? When do they move up or down? . . . left or right? When do they stretch out or shrink?

Two Big Opportunities:

keystrokes: D (for Delete), A (for All), A (for Algebra), A (for Author), type sinx, Enter, C (for Calculus), T (for Taylor), Enter, Enter, Enter, S (for Simplify), Enter, P (for Plot), S (for Scale), hold Del to clear, type 1, Tab, hold Del to clear, type 1, Enter, P (for Plot), and wait until the graph is done, A (for Algebra), A (for Author), type sinx, Enter, P (for Plot), P (for Plot).

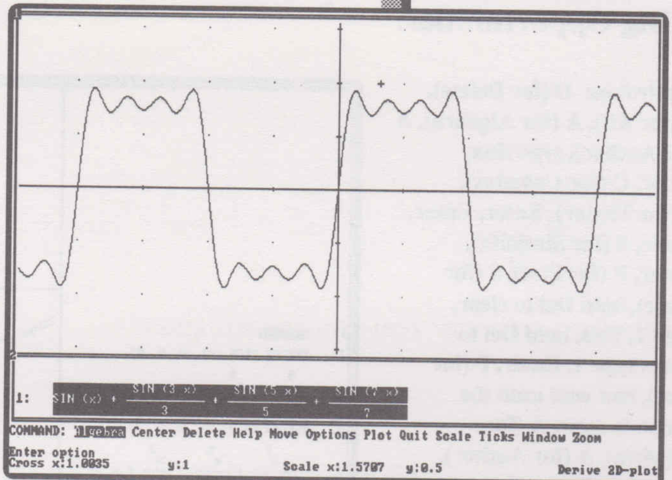


This shows a great idea. A Taylor Expansion creates an approximation for the original function. So, we see that the graph of sin(x) seems equal to the graph of $x^5/120 - x^3/6 + x$, especially around $x = 0$.

keystrokes: A (for Algebra), C (for Calculus), T (for Taylor), Enter, Enter, type 7, Enter, S (for Simplify), Enter, P, P.

This gives an even closer result. Look at these three graphs superimposed and then tap key F10 once and look again.

Try lots of things here . . . cos(x) or tanx instead of sin(x), or e^x, or extend to more terms, or center at some place other than 0. Try 3 factorial (3!) and simplify . . . and 4 factorial (4!) . . . see anything? . . . Go . . . go . . . go.



keystrokes: W (for Window), C (for Close), Enter, R (for Remove), tap Home key, Enter, A (for Author), type sinx+sin(3x)/3+sin(5x)/5+sin(7x)/7, Enter, W (for Window), S (for Split), H (for Horizontal), tap Del to clear, type 16, Enter, W (for Window), D (for Designate), 2 (for 2D–plot), y (for yes), S (for Scale), tap Del to clear, hold Alt key and type P, type /2, Tab, tap Del to clear, type 0.5, Enter, O (for Options), A (for Accuracy), tap Del to clear, type 8, Enter, P (for Plot).

What would this curve look like if it were extended for 5 more terms? Use F9 to move in for a close look. Try sin(x) on top of this and sin(3x)/3 etc. The name of J. Willard Gibbs, a famous American physicist of the 19th century, is connected to this system. There's a lot here to wonder about.

Chapter 11

Solving Trigonometric Equations

At times we are expected to solve equations which have variables inside of trig functions. Derive can solve many such equations by a variety of methods. Suppose our equation is sin(x) = 1/2 and we are asked to locate numerical value(s) for x that make this open sentence true, (or, as many people would say, find value(s) of x to "satisfy the equation").

One approach always available is to substitute some numbers for x to get a "feel" for the situation.

keystrokes: A (for Author), type sinx=1/2, Enter, M (for Manage), S (for Substitute), Enter, type 3, Enter, X (for approXimate), Enter.

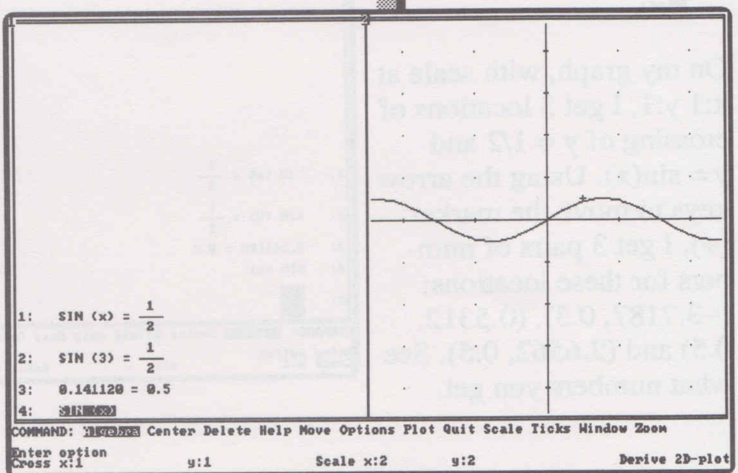
1: $\text{SIN } x = \frac{1}{2}$

2: $\text{SIN } 3 = \frac{1}{2}$

3: 0.141120 = 0.5

Now that we see the result of our first guess, in what direction do we go? My usual idea at this point is to ask "too big or too small?" Since 0.141120 is smaller than 0.5, maybe we need something bigger than 3 for x. If x gets bigger, does the sin(x) get bigger? If we don't know, it's unclear how to proceed. Maybe a graph of sin(x) would help. First we set up a graphing window and then graph the sine.

keystrokes: W (for Window), S (for Split), V (for Vertical), Enter, F1 (to move next window), W (for Window), D (for Designate), type 2 (for 2D plot), y (for yes), A (for Algebra), A (for Author), type sinx, Enter, P (for Plot), S (for Scale), tap Del to clear, type 2, Tab, tap Del to clear, type 2, Enter, P (for Plot).



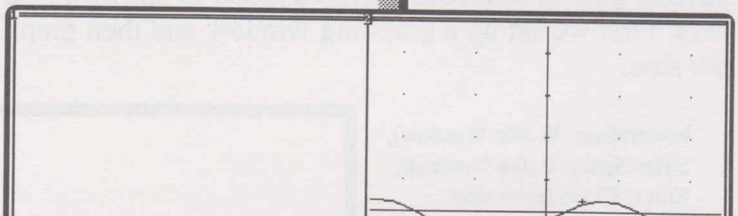
With our graph of the sinx in front of us and the marker (+) to locate points, we are in business. Tap the arrow keys a few times to see the marker (+) move on the screen. Tap PgUp and PgDn for bigger jumps and hold the Ctrl key down and tap arrow left and arrow right for bigger jumps horizontally. Look at the lower left part of your screen to see the x and y readings for where the marker (+) is on the screen.

Move the marker up or down until the y reading is .5 and then left or right until the marker (+) is on the curve. When I do this I get x:2.5937 y:0.5. What do you get? Can we have different answers?

Can we both be correct?

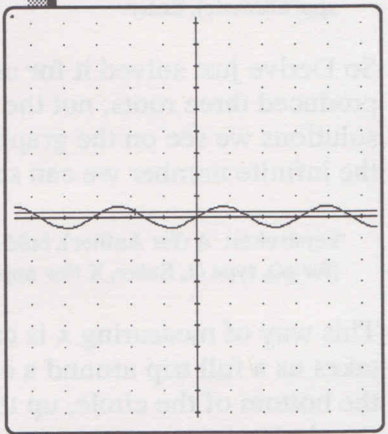
One way to locate the places on the graph of sin(x) where y is 1/2 is by placing a second graph onto the field. What graph? $y = 1/2$.

> **keystrokes: A (for Algebra), A (for Author), type 1/2, Enter, P (for Plot), P (for Plot).**



On my graph, with scale at x:1 y:1, I get 3 locations of crossing of $y = 1/2$ and $y = \sin(x)$. Using the arrow keys to move the marker (+), I get 3 pairs of numbers for these locations: (–3.7187, 0.5), (0.5312, 0.5) and (2.6562, 0.5). See what numbers you get.

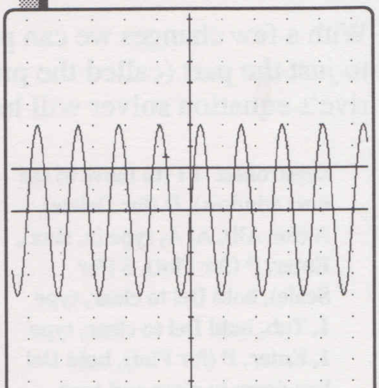
If I now tap F10 (to back away) and see a wider view of the sine curve and y = 1/2, I notice that there are 6 or 7 points of crossing, depending on what monitor I'm using (monochrome graphic monitors show more).



keystrokes: F10 (to move away).

If I do F10 again the picture is hard to see but about 15 or 16 or 17 crossing points appear. In order to see the crossing points more clearly, I change the y scale to 0.5, giving a vertically exaggerated picture but one that serves my purpose.

Your picture and mine may vary a bit because our monitors are different. You can change the scale to get the picture you want.



keystrokes: S (for Scale), tap Del to clear, type 10, Tab, tap Del to clear, type 0.5, Enter.

If we use arrow keys to move on the grid, we can then read off each of the values with some decent accuracy (hold Ctrl key down and tap arrow left and/or arrow right keys for larger jumps). Since sine curves go on forever to the right and left, there are an infinite number of solutions. There is also a regular pattern to where they occur. You may want to try this yourself. The graphical means described above can be used for all kinds of situations both with trig functions and without. Please make yourself a note to use this method in many situations until it becomes a habit. It produces good results. Besides, no other way makes it so clear as to what is really going on.

How else could sin(x) = 1/2 be solved? Well . . . how about a nice simple way?

keystrokes: A (for Algebra), A (for Author), type solve(sinx=1/2, x), Enter, S (for Simplify), Enter, X (for approXimate), Enter.

1: $\text{SOLVE}\left[\text{SIN}(x) = \frac{1}{2}, x\right]$

2: $\left[x = -\frac{7\pi}{6}, x = \frac{\pi}{6}, x = \frac{5\pi}{6}\right]$

3: [x = -3.66519, x = 0.523598, x = 2.61799]

So Derive just solved it for us. It produced three roots, not the 16 solutions we see on the graphs or the infinite number we can see in our minds.

keystrokes: A (for Author), hold down the Alt key and tap P (for pi), type /2, Enter, X (for approXimate), Enter.

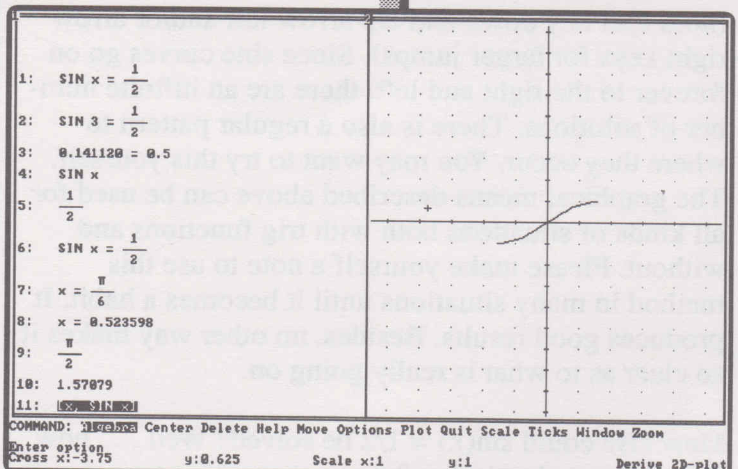
9: $\frac{\pi}{2}$

10: 1.57079

This way of measuring x is called radians; 2π radians takes us a full trip around a circle. $-\pi/2$ to $\pi/2$ is from the bottom of the circle, up the right side, to the top of the circle.

With a few changes we can produce a sine curve limited to just the part (called the principal value), where Derive's equation solver will look for a solution.

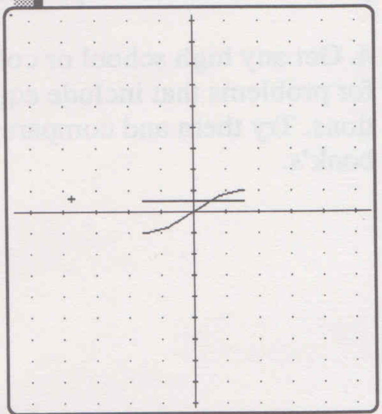
keystrokes: F1 (to move to the next window), D (for Delete), A (for All), A, A, type [x, sinx], Enter, P (for Plot), S (for Scale), hold Del to clear, type 1, Tab, hold Del to clear, type 1, Enter, P (for Plot), hold Del key down to clear and type −1.57, Tab, hold Del key down to clear and type 1.57, Enter.



This shows the sine curve from $-\pi/2$ to $\pi/2$. We accomplished this by using the parametric form (that is, the square brackets

notation and giving left and right boundaries to the variable). We'll now pick up the straight line by parametrics.

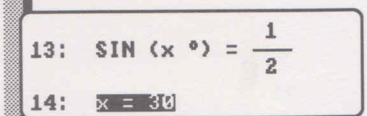
keystrokes: A (for Algebra), A (for Author), type [x, 1/2], Enter, P (for Plot), P (for Plot), Enter.



So we can see there is only one solution in the restricted domain of the principal value of sine.

The solution obtained by solving sin(x) = 1/2 algebraically was π/6 or .523598. We can also get the answer in degrees.

keystrokes: A (for Algebra), A (for Author), type sin(xdeg) = 1/2, Enter, L (for Solve), Enter.



We see that sin(30 degrees) and sin(pi/6) are the same number; it's like saying 2^3 and 8 . . . different names for the same number.

Let me leave you with a list of Primo Opportunities for our further development.

Primo Opportunities:

1. Graph $\sin(x^2)$ and y = 1/2 and look for where they cross. Compare these results with the results in the main problem of the chapter.

2. Graph $(\sin(x))^2$ and y = 1/4 or $(\sin(x))^3$ and y = 1/8 and look for crossing points. Try it algebraically.

3. Solve sin(2xdeg) = 1/2 and sin(x) = 1/2 . . . compare your results. Try solving sin(2xdeg) = 1 and sin(x) = 1 and compare. Try sin(9x) = 1 and sinx = 1. What do you see? . . . what did you expect?

4. Get any high school or college algebra book and look for problems that include equations involving trig functions. Try them and compare Derive's results with the book's.

Chapter 12

Subtleties of Derive

Subtle Situations

Gradually, over the past three and one–half years of using Derive, I have learned that A Mathematical Assistant is not only powerful, but also subtle. I recommend this chapter especially for harried reviewers of computer programs, who don't have the opportunity to use a program long enough to see what you will see here.

For example, Derive handles the absolute value function with great care.

> **keystrokes: A (for Author), type absx, Enter, S (for Simplify), Enter.**

```
1:   |x|
2:   |x|
```

We see that when we simplify the absolute value of x, Derive returns the same expression we put in. We all need to realize that this does not necessarily mean that Derive cannot deal with the situation. Derive is simply exercising appropriate restraint. I'm using x in the default case, which is a real number defined from negative infinity to positive infinity.

What if we choose a specific value for x?

> **keystrokes: A (for Author), type abs4, Enter, S (for Simplify), Enter.**

```
3:   |4|
4:   4
```

Here we have a tiny bit of action. The absolute value of 4 is just plain 4. What about the absolute value of a negative number?

> **keystrokes: A (for Author), type abs(–7), Enter, S (for Simplify), Enter.**

```
5:   |-7|
6:   7
```

It seems that the absolute value of a number is the positive number with the same digits.

Now to the subtle part. When we authored absx and simplified, Derive returned what we authored in an unchanged form. Suppose we Declare x to be positive and then simplify absx and then repeat the process with x declared to be negative. What will happen?

keystrokes: D (for Declare), V (for Variable), type x, Enter, P (for Positive), arrowup (to highlight absx), S (for Simplify), Enter.

```
7:     x
```

So we see that the absolute value of x is x itself as long as x is positive.

What if x is negative? Will Derive simplify abs(x) if x is negative?

keystrokes: D (for Declare), V (for Variable), type x, Enter, R (for Real), tap Del key to clear, type –inf, tab 3 times, tap Del key to clear, type 0, Enter, arrowup (to highlight absx), S (for Simplify), Enter.

```
8:     - x
```

We see that abs(x) is –x if x itself is a negative number. People ask how can abs(x) be a negative, namely –x? Well, –x is not a negative number if x is a negative number. Try the following to see what's happening.

keystrokes: Arrowup (to highlight absx), S (for Simplify), Enter, M (for Manage), S (for Substitute), Enter, type –3, Enter, S (for Simplify), Enter.

```
9:     - x
10:    - -3
11:    3
```

If x is positive then abs(x) is just plain x. BUT if x is negative, abs(x) is –x; which says, Take x (which is negative) and take the opposite of it.

Opportunities:

1. Find abs(x^3) simplified if x is Declared positive.

2. Find abs(x^3) simplified if x is Declared negative.

3. Find abs(x^4) simplified if x is Declared positive.

4. Find abs(x^4) simplified if x is Declared negative.

5. Find abs(x^{13}) simplified if x is Declared positive.

6. Find abs(x^{13}) simplified if x is Declared negative.

Another example of wonderful software design is Derive's handling of inequalities. If $2x < 12$ then $x < 6$.

keystrokes: A (for Author), type 2x<12, Enter, L (for soLve), Enter.

```
12:   2 x < 12
13:   x < 6
```

No problems in sight! But . . .

keystrokes: A (for Author), type –2x<12, Enter, L (for soLve), Enter.

```
14:   - 2 x < 12
15:   x > -6
```

Did you notice in the last case that the inequality sign changed from < to >? Again, it is a case of handling positive and negative values with care. If we generalize with k, a real number from negative infinity to positive infinity, how will Derive handle $kx < 12$?

keystrokes: A (for Author), type kx<12, Enter, L (for soLve), Enter, Enter.

```
16:   k x < 12
17:   k x < 12
```

Derive takes no action in solving $kx < 12$. It seems ineffective.

Let's Declare k to be positive and try again.

keystrokes: D (for Declare), V (for Variable), type k, Enter, D, (for Domain), P (for Positive), L (for soLve), Enter, Enter.

18: $x < \dfrac{12}{k}$

And what if we Declare k to be negative? How will Derive react?

keystrokes: D (for Declare), V (for Variable), type k, Enter, R (for Real), tap Del key to clear, type –inf, Tab, Tab, Tab, type 0, Enter, arrowup (to highlight kx<12), L (for soLve), Enter, Enter.

19: $x > \dfrac{12}{k}$

I believe this is an impressive display of proper use of mathematical symbols; a quality which deserves recognition and praise.

When I first began to use symbolic algebra programs, I was amazed to find that ab was read as a single variable and not as the product of two variables a and b. This seemed absurd and caused me to wonder about such programs. I complained to people more experienced than I and they said it had to be this way. People want to use a common word as a variable, like force and don't want it to be interpreted as f*o*r*c*e. I saw their concern but I wasn't happy. I often teach people beginning algebra and I thought this procedure would make algebra appear confusing for the naive user. How does Derive deal with this situation? Beautifully!

keystrokes: A (for Author), type (force)^2, Enter, S (for Simplify), Enter.

20: $(f\ o\ r\ c\ e)^2$

21: $c^2\ e^2\ f^2\ o^2\ r^2$

In the default case force is seen as f*o*r*c*e so when squared and simplified each letter is a variable and squared. This is not what you want for the variable force but it is what we want for ab, that is, a*b.

keystrokes: A (for Author), type (ab)^2, Enter, S (for Simplify), Enter.

22: $(a\ b)^2$

23: $a^2\ b^2$

Here are a few examples in the Word Mode:

keystrokes: A (for Author), type force=mass*acceleration, Enter, L (for soLve), Enter, hold down the Del Key to clear, type mass, Enter.

26: $\text{force} = \text{mass acceleration}$

27: $\text{mass} = \dfrac{\text{force}}{\text{acceleration}}$

keystrokes: A (for Author), type (p1*v1)/t1=(p2*v2)/t2, Enter, L (for soLve), Enter, tap Del to clear, type t2, Enter.

28: $\dfrac{p1\ v1}{t1} = \dfrac{p2\ v2}{t2}$

29: $t2 = \dfrac{p2\ t1\ v2}{p1\ v1}$

p, v and t represent pressure, volume and temperature of a gas.

Many of us expect the cube root of −8 to be −2 since $(-2)^3 = -8$. Derive thinks the answer is 1+sqrt(3)î because $(1+\text{sqrt}(3)\hat{\imath})^3$ is −8. Mathematicians say this is the "correct solution" because it is the Principal Value in the complex plane. We can get the "common sense" answer by using Manage-Branch-Real to get the cube root of −8 = −2. If your graph of $y = x^{1/3}$ is less than you expect, do Manage-Branch-Real and graph again.

Here are three very different circumstances in which Derive responds with no action (echoes your input):

1. Request the inverse of a matrix whose determinant is zero. In this case Derive is signaling that there is no correct answer.

2. Simplifying sin(2x) with all settings at default. In this case Derive is signaling that it cannot do the job; if we do Manage-Trig-Expand-Enter-Simplify, we get our result. So in this case there is an answer which Derive can obtain but we must make adjustments.

3. Ask Derive to solve sin(2x)=cos(3x) and we get no answer except a message "Memory Full". This indicates that Derive cannot do it and we don't know why (it might be a bug). If we do Manage-Trig-Expand-Enter-Simplify and then soLve we get our results.

Another wonderful (but confusing) example of Derive sublety is:

keystrokes: A (for Author), type x^n, Enter, C (for Calculus), I (for Integrate), Enter, Enter, Enter, S (for Simplify), Enter

1: x^n

2: $\int x^n \, dx$

3: $\dfrac{x^{n+1} - 1}{n + 1}$

This looks familiar but wrong. Why isn't the result of the integral $x^{n+1}/n+1$? The answer is: we are in trouble if n is -1. When $n = -1$ the simplification of the integral should be ln(x). Derive is taking this fact into consideration and presents one answer which is true in all cases.

keystrokes: C (for Calculus), L (for Limit), Enter, type n, Enter, type -1, Enter, S (for Simplify), Enter

4: $\lim\limits_{n \to -1} \dfrac{x^{n+1} - 1}{n + 1}$

5: LN (x)

If we declare n to be positive and simplify the same integral we get the usual answer.

I know of no other symbolic algebra program which exhibits this extreme degree of care. Bravo!

Chapter 13

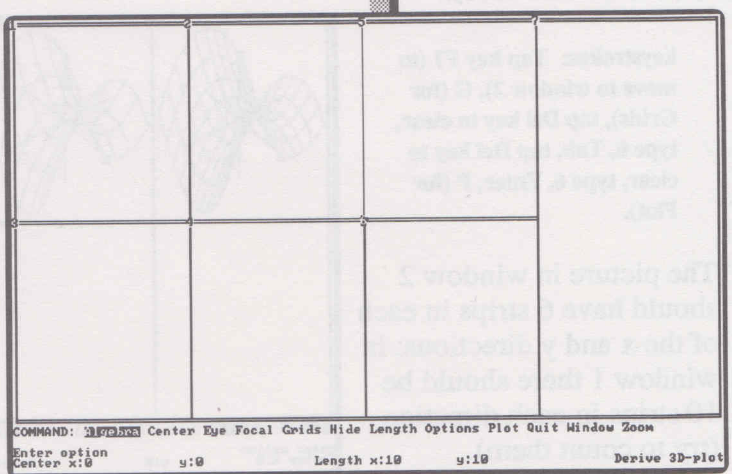
3D Graphs

I'll start you off on 3D graphs quickly by suggesting that we split up our screens into 7 windows: 1 full length window, at the right side of our screen for the equations that we'll graph, and 6 approximately square windows in 2 rows of 3 each which will be 3D windows. What I'll describe applies particularly to a system with EGA or VGA color; you'll face fewer choices and have less pretty results in monochrome or in CGA. First, we'll prepare the conditions for the single window.

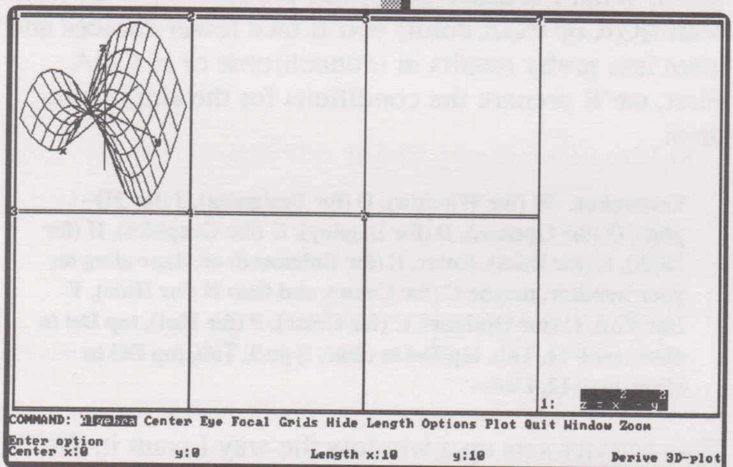
keystrokes: W (for Window), D (for Designate), 3 (for 3D–plot), O (for Options), D (for Display), G (for Graphics), H (for High), E (for EGA), Enter, E (for Enhanced) or, depending on your monitor, maybe C (for Color), and then H (for Hide), Y (for Yes), O (for Options), C (for Color), P (for Plot), tap Del to clear, type 11, Tab, tap Del to clear, type 9, Tab, tap Del to clear, type 12, Enter.

This activity sets up a window the way I want it. We'll next split this window into several, all with the same characteristics as the first. We will not have to adjust each window.

keystrokes: W (for Window), S (for Split), V (for Vertical), Enter, W (for Window), S (for Split), H (for Horizontal), Enter, W (for Window), S (for Split), V (for Vertical), Enter, tap key F1 twice (to move to window 3), W, S, V, Enter, tap key F1 twice (to move to window 5), W, S, V, Enter, W, S, H (for Horizontal), Enter, tap key F1 twice (to move to window 7), W, D (for Designate), A (for Algebra).



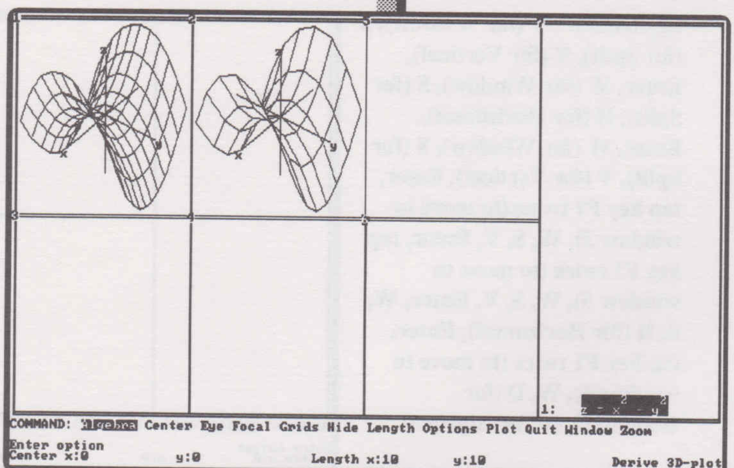
Now we have 6 graphing windows and 1 long algebra window. We will create similar pictures in each graphing window to show some of the different possibilities for 3D graphing with Derive. Tapping key F1 will move you forward through the windows; to move back through the same windows, hold down Ctrl (or Control) key and tap F1. Try it now.



Next we'll enter an equation to plot, accepting the initial conditions of all the controlling variables.

> **keystrokes: A (for Algebra – if you're not already in the Algebra window), A (for Author), type z=x^2–y^2, Enter, tap key F1 (to move to window 1), P (for Plot).**

We see a 3D plot that is usually called a saddle. For our first change we'll make the grid numbers 6 (instead of 10, the default case).

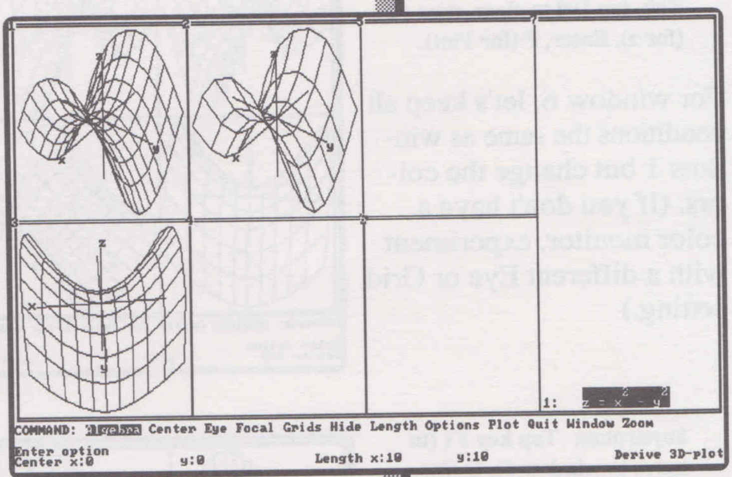


> **keystrokes: Tap key F1 (to move to window 2), G (for Grids), tap Del key to clear, type 6, Tab, tap Del key to clear, type 6, Enter, P (for Plot).**

The picture in window 2 should have 6 strips in each of the x and y directions; in window 1 there should be 10 strips in each direction (try to count them).

Next we'll go to window 3 and change the location of the "Eye" that looks at the 3D graph, thus changing our view. In windows 1 and 2, the saddle is viewed from x=20, y=15 (the default settings), and z=75 (a position computed by Derive). In window 3 we'll look from x=1, y=15 and z=75. Before you plot the graph try to imagine what the saddle will look like.

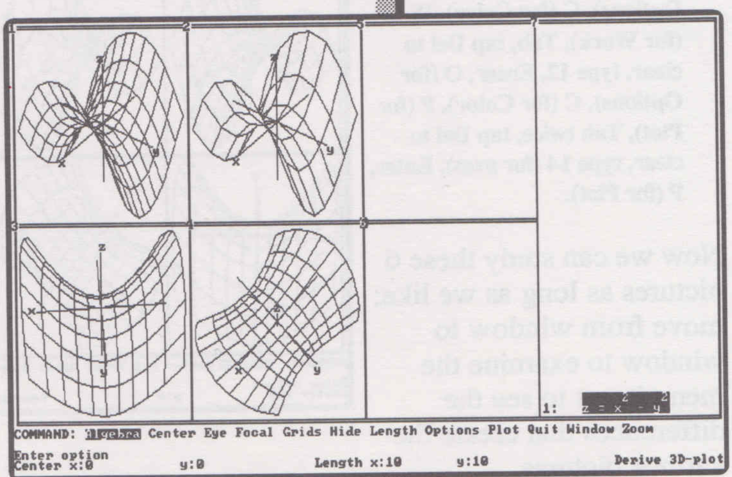
keystrokes: tap key F1 (to move to window 3), E (for Eye), tap Del key to clear, type 1 (for x), Enter, P (for Plot).



In window 3, for the first time, we can easily count the 10 strips in the x direction.

For window 4, we'll look from x=1, y=1 and z=75. Imagine the result.

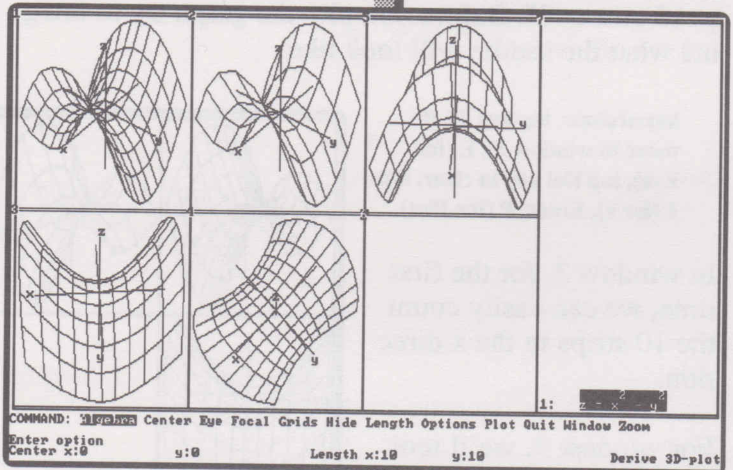
keystrokes: tap key F1 (to move to window 4), E (for Eye), tap Del to clear, type 1 (for x), Tab, tap Del to clear, type 1 (for y), Enter, P (for Plot).



In window 4, we can see clearly the 10 strips in each of the x and y direction.

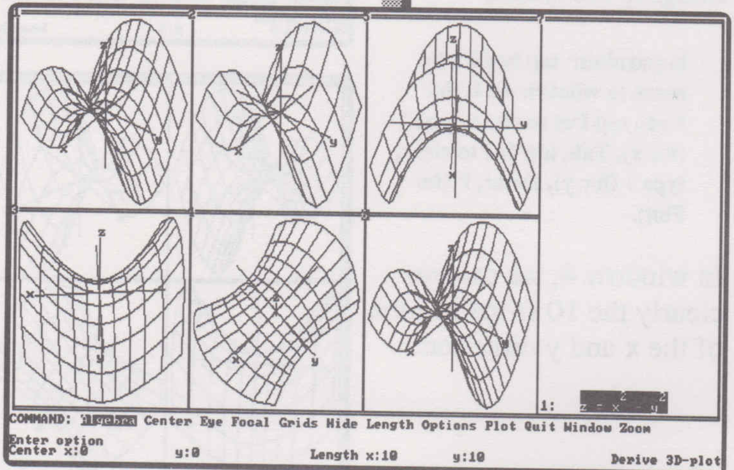
Next, how about a look at the saddle from an end, where we can see a bit of the underside? Think of a set of Eye coordinates that would do this. Then see what I chose.

keystrokes: F1 (to move to window 5), E (for Eye), Tab, Tap Del to clear, type 1 (for y), Tab, tap Del to clear, type 45 (for z), Enter, P (for Plot).



For window 6, let's keep all conditions the same as window 1 but change the colors. (If you don't have a color monitor, experiment with a different Eye or Grid setting.)

keystrokes: Tap key F1 (to move to window 6), O (for Options), C (for Color), W (for Work), Tab, tap Del to clear, type 12, Enter, O (for Options), C (for Color), P (for Plot), Tab twice, tap Del to clear, type 14 (for axes), Enter, P (for Plot).



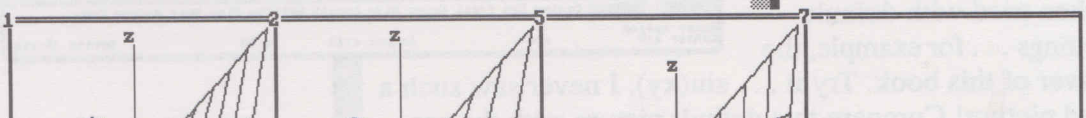
Now we can study these 6 pictures as long as we like; move from window to window to examine the menu items to see the differences that create the various pictures.

When you are ready, we can try a new equation and run it through the 6 windows.

> **keystrokes: A (for Algebra), A (for Author), type z=x^4*y, Enter, tap F1 (to move to window 1), O (for Options), C (for Color), P (for Plot), Tab twice, tap Del to clear, type 12 or some other color you'd prefer (for axes), Enter, P (for Plot).**

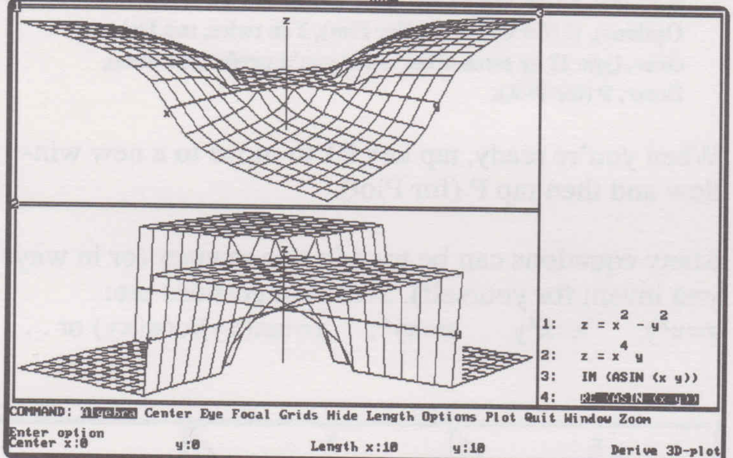
When you're ready, tap key F1 to move to a new window and then tap P (for Plot).

Many equations can be tried in this manner (or in ways you invent for yourself). Some suggestions are:
$z=x^2y$, $z=x^4y$, $z=xy^4$, $z=\tan(xy)/\cos(xy)$ or . . .

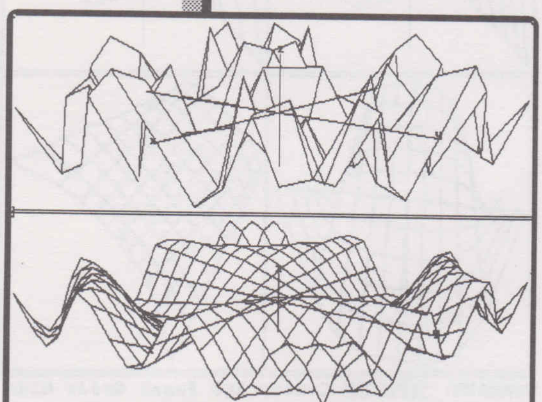


A function like arc sine of xy will generate complex numbers for some values of xy. If we plot the imaginary part we could get an interesting graph.

keystrokes: Use the F1 key to move to window 6, W (for Window), C (for Close), repeat 3 more times leaving two 3D windows and 1 algebra window, A (for Algebra), A (for Author), type im(asin(xy)), Enter, P (for Plot), P (for Plot) and wait. Then try re(asin(xy)) in the other 3D window.



One last word of warning: Many 3D graphs look awful when tried with default settings . . . for example, the cover of this book. Try it . . . sin(xy). I never saw such a bad picture! Compare this default picture with the second picture. It's hard to believe they are generated by the same expression.



How do we improve the picture? The length is shortened to 2π in the x and y direction, the z is shortened to about 2, gridlines are changed from 10 to about 20 (for super resolution, as on the cover, set them to 40) and the location of the Eye in the z direction is raised to about 8. Also the color on the top is changed to 11 (light blue) and the color on the bottom is changed to 9 (dark blue). The trade–off in changing the settings is that the graph takes longer to calculate and appear on your screen.

Chapter 14

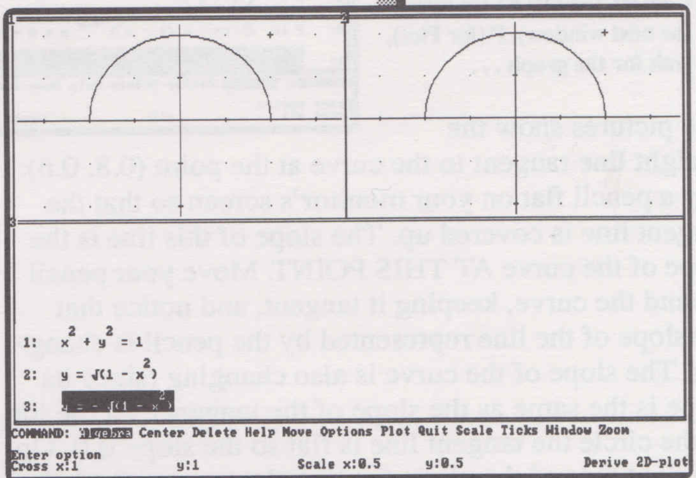
Calculus

Calculus was invented to solve problems that involve change. If the world were static, calculus might not have been invented.

Many problems in the world that involve change can be reduced to two problems: find the slope of a curve at any point, and find the area of a region that is bounded by at least one curve. First I will consider the slope question.

If we look at the graph of the curve $x^2+y^2=1$ we see a circle of radius 1. I'll show you how to find the slope at a point, (0.8, 0.6), on this unit circle.

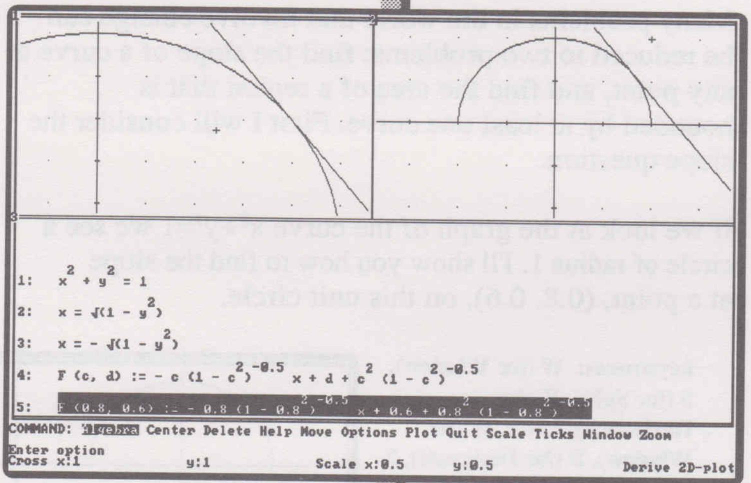
keystrokes: W (for Window), S (for Split), H (for Horizontal), Enter, W (for Window), D (for Designate), 2 (for 2D–plot), W (for Window), S (for Split), V (for Vertical), Enter, A (for Algebra), A (for Author), type x^2+y^2=1, Enter, L (for Solve), Enter, tap Del to clear, type y, Enter, arrowup (to highlight sqrt(1–x²)), P (for Plot), P (for Plot), tap key F1 (to move to the next window), P (for Plot).



This gives the top half of the circle. (It's not necessary to plot the bottom since the point we are aiming for, (0.8, 0.6), is on the top.) I have chosen a circle for our demonstration because it has a nice sharp bend and we can easily see the difference between the curve and the straight line tangents and approximations to tangents.

To see the tangents to a curve, I have written a little function that we will use without explanation of how it was created.

keystrokes: A (for Algebra), A (for Author), type f(c,d):= –c(1–c^2)^–0.5x+d+c^2 (1–c^2)^–0.5, Enter, M(for Manage), S (for Substitute), Enter, Enter,type 0.8, Enter, type 0.6, Enter, P (for Plot), P (for Plot), wait for the graph . . ., PgDn, hold Ctrl key down and tap arrowleft, tap key F9 (to move in), C (for Center), tap key F1 (to move to the next window), P (for Plot), wait for the graph . . .



The pictures show the straight line tangent to the curve at the point (0.8, 0.6). Lay a pencil flat on your monitor's screen so that the tangent line is covered up. The slope of this line is the slope of the curve AT THIS POINT. Move your pencil around the curve, keeping it tangent, and notice that the slope of the line represented by the pencil is changing. The slope of the curve is also changing (since its slope is the same as the slope of the tangent). At the top of the circle the tangent line is flat so the slope is 0. On the right side of the circle the line slants up to the left so the slope is negative. At the left side of the picture the line slants up to the right so the slope is positive. Use the F(c,d) function to create lines tangent to the circle. C and d represent the x and y values of points on the circle; you may want to return to the y = sqrt(1 – x^2) line so that you can substitute a value for x and let Derive calculate the corresponding value for y.

How do we understand getting the slope at (0.8, 0.6) of the circle? We'll get the slope of straight lines that intersect the circle at (0, 1) and (0.8, 0.6) and (0.6, 0.8) and (0.8, 0.6) to show a trend toward the one we want. We'll do this graphically and then numerically. Again, I'll introduce a function, without explanation, that will draw the lines we want.

keystrokes: A (for Algebra), A (for Author), type line(g,h,i,j):=(h–j)/(g–i)x+h–(h–j)/(g–i)g, Enter,

$$6: \quad \mathrm{LINE}(g, h, i, j) := \frac{h - j}{g - i} x + h - \frac{h - j}{g - i} g$$

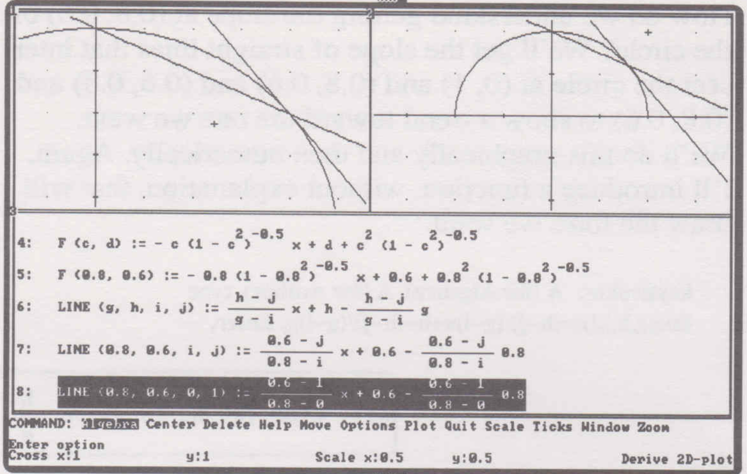
This function, when plotted, creates a line connecting the points(g, h) and (i, j). We're interested in lines that connect (0.8, 0.6) and some other point, so we'll make that substitution right away.

keystrokes: M (for Manage), S (for Substitute), Enter, Enter, type 0.8, Enter, type 0.6, Enter, Enter, Enter.

$$7: \quad \mathrm{LINE}(0.8, 0.6, i, j) := \frac{0.6 - j}{0.8 - i} x + 0.6 - \frac{0.6 - j}{0.8 - i} 0.8$$

We are now free to concentrate on what our second point will be. I suggest (0, 1) to start.

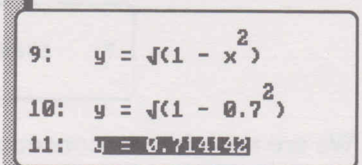
keystroke: M (for Manage), S (for Substitute), Enter, Enter, type 0, Enter, type 1, Enter, P (for Plot), P (for Plot).



We see the line connecting (0, 1) on the circle and (0.8, 0.6) on the circle. The slope of this line can be thought of as an approximation to the slope of the tangents to the curve at (0.8, 0.6). By choosing points closer to (0.8, 0.6) we can get a better approximation. Try it yourself. Find points on the circle that get closer to the target of (0.8, 0.6), substitute those numbers into the line function, and plot.

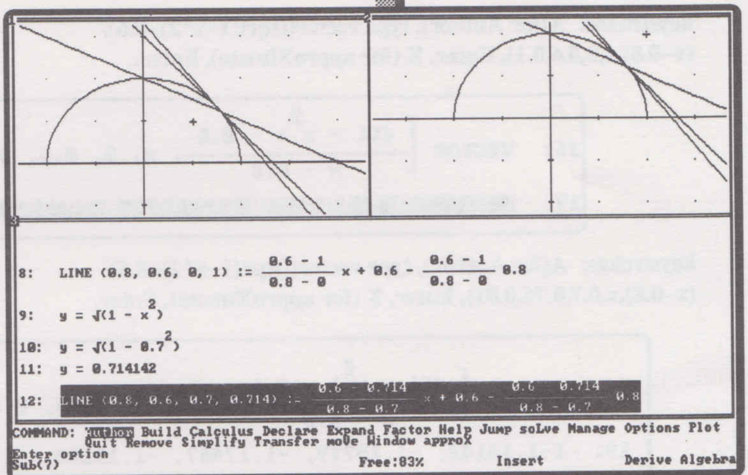
I'll find two numbers for (i, j) that are very close to (0.8, 0.6) and use my line function to plot a graph.

keystrokes: A (for Algebra), arrowup (to highlight y=sqrt(1−x²)), M (for Manage), S (for Substitute), Enter, type 0.7 (a number close to 0.8), Enter, Enter, X (for approXimate), Enter.



When x is 0.7, a corresponding y value ON THE CIRCLE is 0.714. We'll now substitute these two values for i and j in the LINE function, and plot.

keystrokes: Arrowup (to highlight LINE(0.8,0.6,i,j):= etc.), M(for Manage), S (for Substitute), Enter, type 0.7, Enter, typ e 0.714, Enter, P (for Plot), P (for Plot), wait for the graph . . . , tap key F1, P (for Plot).



Notice how similar the last plot, which is simply a line connecting two points on the circle (no calculus), is to the tangent line at a single point, nearby, that was created by calculus. The new idea (calculus) grows out of the old idea.

How can we see the numerical value of the slope of the tangent line at (0.8, 0.6)? First, we'll define the slope as the change in y value divided by the corresponding change in the x value. Since the line whose slope we want is from (x, y) on the circle to (0.8, 0.6) also on the circle, the slope is:

keystrokes: A (for Algebra), A (for Author), type slope:=(sqrt(1–x^2)–0.6)/(x–0.8), Enter, M (for Manage), S (for Substitute), Enter, type 0, Enter, X (for approXimate), Enter.

13: $\text{SLOPE} := \dfrac{\sqrt{(1 - x^2)} - 0.6}{x - 0.8}$

14: $\text{SLOPE} := \dfrac{\sqrt{(1 - 0^2)} - 0.6}{0 - 0.8}$

15: -0.5

We can now calculate the value of the slope at a variety of points, moving closer and closer to our target spot of (0.8, 0.6), using the vector command.

keystrokes: A(for Author), type vector((sqrt(1–x^2)–0.6)/(x–0.8),x,0,0.6,0.1), Enter, X (for approXimate), Enter.

$$16:\quad \text{VECTOR}\left[\frac{\sqrt{(1-x^2)}-0.6}{x-0.8},\ x,\ 0,\ 0.6,\ 0.1\right]$$

$$17:\quad [-0.5,\ -0.564267,\ -0.632993,\ -0.707878,\ -0.791287,\ -0.886751,\ -1]$$

keystrokes: A(for Author), type vector((sqrt(1–x^2)–0.6)/(x–0.8),x,0.7,0.76,0.01), Enter, X (for approXimate), Enter.

$$18:\quad \text{VECTOR}\left[\frac{\sqrt{(1-x^2)}-0.6}{x-0.8},\ x,\ 0.7,\ 0.76,\ 0.01\right]$$

$$19:\quad [-1.14142,\ -1.15779,\ -1.17467,\ -1.19210,\ -1.21011,\ -1.22875,\ -1.24807]$$

keystrokes: A(for Author), type vector((sqrt(1–x^2)–0.6)/(x–0.8),x,0.7999,0.79996,0.00001), Enter, O (for Options), P (for Precision),Tab, tap Del to clear, type 12, Enter, X (for approXimate), Enter.

$$20:\quad \text{VECTOR}\left[\frac{\sqrt{(1-x^2)}-0.6}{x-0.8},\ x,\ 0.7999,\ 0.79996,\ 0.00001\right]$$

$$21:\quad [-1.33310190328,\ -1.33312504165,\ -1.33314818106,\ -1.33317132145,\ -1.33319446296,\ -1$$

To calculate the slope a bit more directly we'll use Derive's ability with limits. We'll find the limit of the slope expression as the variable approaches 0.8.

keystrokes: A (for Author), type (sqrt(1–x^2)–0.6)/(x–0.8), Enter, C (for Calculus), L (for Limit), Enter, Enter, tap Del to clear, type 0.8, Tab, B (for Below), Enter, S (for Simplify), Enter, X (for approXimate), Enter.

$$22:\quad \frac{\sqrt{(1-x^2)}-0.6}{x-0.8}$$

$$23:\quad \lim_{x\to 0.8-}\frac{\sqrt{(1-x^2)}-0.6}{x-0.8}$$

$$24:\quad -\frac{4}{3}$$

$$25:\quad -1.33333333333$$

Or we can simply do the calculus directly.

keystrokes: A (for Author), type sqrt(1–x^2), Enter, C (for Calculus), D (for Differentiate), Enter, Enter, Enter, S (for Simplify), Enter, M (for Manage), S (for Substitute), Enter, type 0.8, Enter, S (for Simplify), Enter, X (for approXimate), Enter.

25: $\sqrt{(1 - x^2)}$

26: $\frac{d}{dx}\sqrt{(1 - x^2)}$

27: $-\frac{x}{\sqrt{(1 - x^2)}}$

28: $-\frac{0.8}{\sqrt{(1 - 0.8^2)}}$

29: $-\frac{4}{3}$

30: -1.33333333333

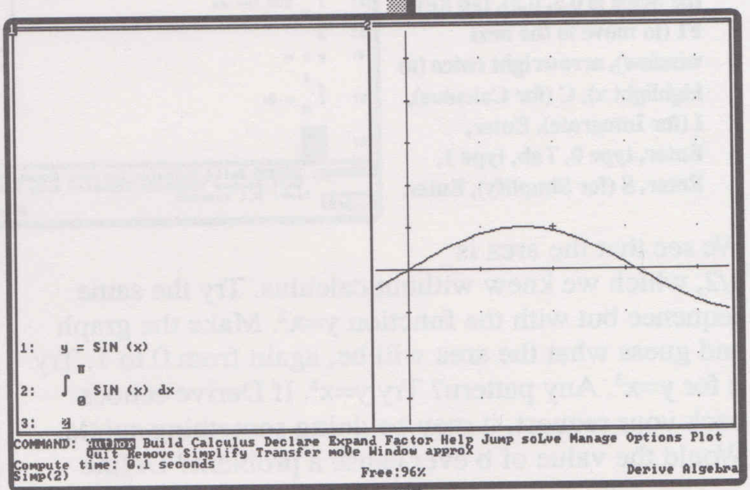
Usually, learning the integral calculus begins with a basic development of area via limits. Instead, I'd like to explore some intuitively attractive ideas using Derive's integrating capabilities.

What is the area enclosed by a sine curve and the x axis between 0 and π? Let's look at the graph:

keystrokes: A (for Author), type y = sinx, Enter, W (for Window), S (for Split), V (for Vertical), Enter, tap key F1(to move to the next window), W (for Window), D (for Designate), type 2 (for 2D-plot), y (for yes), P (for Plot), hold down the Control key and tap arrowright twice, C (for Center).

We know that a sine curve rises to a height of 1 and we are measuring the area across a distance of π. Make an estimate and then we'll compute the actual area using calculus.

keystrokes: A (for Algebra), arrowright twice (to highlight sinx), C (for Calculus), I (for Integrate), Enter, Enter, type 0, Tab, hold down the Alt key and type p (for π), Enter, S (for Simplify), Enter.



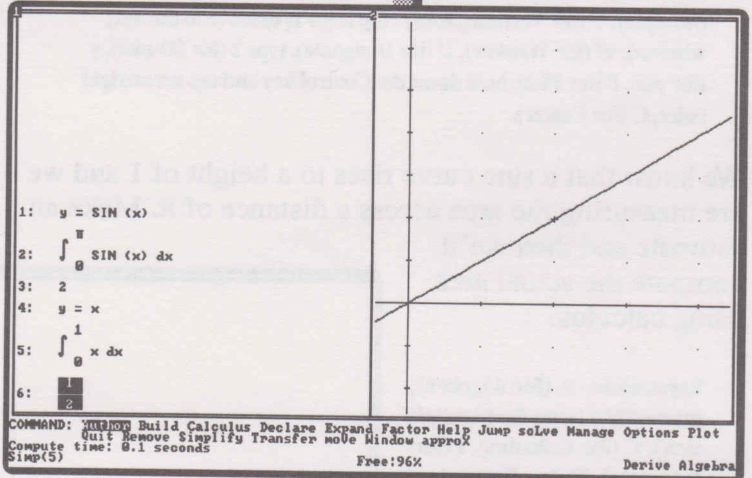
If the area bounded by a sine curve and the x axis from 0 to π is 2, what will be the area enclosed by

2*sin(x)? 3*sin(x)? b*sin(x)? What is the area bounded by a sine curve and the x axis from 0 to 2π? From π to 2π?

Try the same kinds of questions on cosine curves. Be sure to look at the appropriate graphs so you can give your intuition a chance. Try calculating the area of sin(x) from π to 0. What do you expect? Try it!

Let's try another pattern. If we graph y=x and look at the area enclosed by that line and the x and y axes we see a nice isosceles triangle. One height is 1 and the corresponding base is 1, so the area of the triangle is 1/2.

keystrokes: A (for Author), type y=x, Enter, P (for Plot), D (for Delete), A (for All), P (for Plot), wait for the graph to finish, with the arrow keys move the marker(+) to the point (1,1), C (for Center), tap key F9 (to move in so that the Scale is 0.5, 0.5), tap key F1 (to move to the next window), arrowright twice (to highlight x), C (for Calculus), I (for Integrate), Enter, Enter, type 0, Tab, type 1, Enter, S (for Simplify), Enter.



We see that the area is 1/2, which we knew without calculus. Try the same sequence but with the function $y=x^2$. Make the graph and guess what the area will be, again from 0 to 1. Try it for $y=x^3$. Any pattern? Try $y=x^b$. If Derive echoes back your request, it may be doing something subtle. Would the value of b ever cause a problem? Define

variable b to be positive and then try the integration a second time. You might then try to find a generalization by integrating nx^b from 0 to 1. Take a guess at the result.

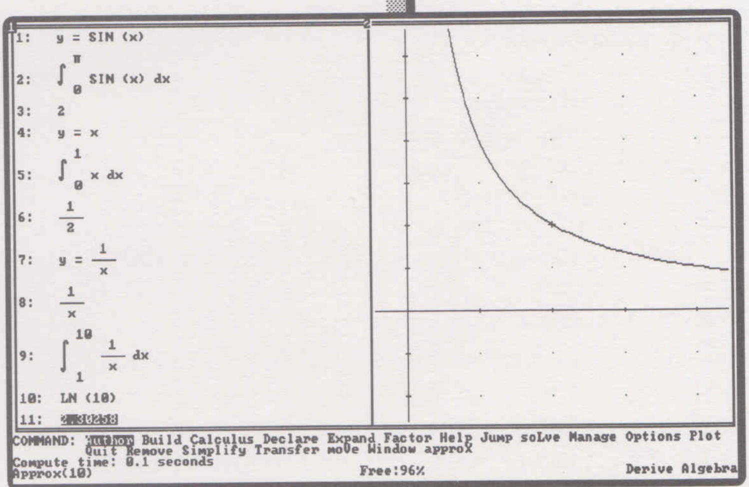
Each of the areas above has been calculated from one finite location to another. People have tried to calcu– late areas across infinite distances (and sometimes have produced finite answers). If we graph y=1/x and just look at the right branch of the curve we may ask ourselves what is the area enclosed by y=1/x from x=1 to x=infinity?

keystrokes: A (for Author), type y=1/x, Enter, P (for Plot), D (for Delete), A (for All), P (for Plot).

Picture a vertical wall at x =1 as the left side of our region and the x axis as the floor and the curve as the top. What is the area enclosed from 1 to 10? 100? 1000? to infinity?

keystrokes: A (for Algebra), A (for Author), type 1/x, Enter, C (for Calculus), I (for Integrate), Enter, Enter, type 1, Tab, type 10, Enter, S (for Simplify), Enter, X (for approXimate), Enter.

Try this same sequence with the other limits. Look at the curve and try to guess the results. Type inf for infinity.



Try this same sequence with the expression $1/x^2$, and $1/x^3$ and $1/x^4$. Keep guessing the result. Try using $1/x^b$ as the expression. If Derive echoes then define b to be real and bigger than 1 and then integrate again.

If you continue to experiment with area you will raise many interesting questions and even answer a few. Have fun!

Using Derive to Teach Math

One of my reasons for thinking that Derive is a wonderful aid for teaching is that it is very powerful. Derive runs on small inexpensive computers and also on high–powered expensive machines. A modest investment will allow people to carry on wonderful investigations.

Because of the easy–to–use Derive interface it's possible for a wide range of students to use exactly the same program. Wouldn't it be nice if a student could start using a program in the 3rd grade and still be using that program in a productive manner 10 years later? What a benefit to not have to learn a new system every year or two but rather to learn new content with the help of the same program!

Another feature of Derive is that it does nothing unless the user puts in a problem to work on. It's like a hammer . . . a tool that needs direction. It is not a complete system of teaching math. For serious benefits to occur, some good teaching is needed. I believe that many teachers will find Derive to be a useful tool to have available for themselves and their students.

How I've used Derive with first and second graders:

Press A (for Author), type 2x+3x and press Enter. Good, good.

Do you know how much 2x and 3x is? Suppose 2 gorillas lived in your house and 3 more moved in? How many gorillas? So, 2x+3x would be . . . yes, 5x.

Let's have the machine do it and see what it gets. Type S for Simplify and tap the Enter key. OK, so 2x and 3x is 5x.

How about 4x+2x? Yes, you could subtract . . . Try it! Sure, whatever numbers you like . . . try 3y+5y or 12g−9g or 2p+3p+4p or 100k+200k−100k+200k or 1+2+3+4+5+6 or . . .

Don't worry about the beep. It means you've done something awful and the whole world will end soon as a result, including your brother.

Ask Marie when you have trouble. Ask Shaun if you have trouble (Shaun is a few years older and a few hours ahead on Derive).

How I've used Derive with first year calculus students.

Derive will do many of the problems you'll face in the next few years in math and in physics and engineering. However, you may not learn anything from the experience except how to type or which command to use in what situation. Also, Derive will be of less help with word problems than with anything else and since everyone has trouble with word problems we'll have to watch for that.

A lot of ideas of calculus connect to limits, such as, what happens to x^2 if x gets close to 3? What do you think? What happens to x^2 if x gets close to 3? . . . sure, x^2 gets close to 9. Would it matter if we approached 3 from below or above? . . . would x^2 still approach 9? Right!

Let's see if Derive can do this same problem. Tap A (for Author), type x^2 and tap Enter. Good . . . good. Look at the menu . . . notice the capital letter in each word? Tap C (for Calculus), L (for Limits), Enter and Enter, which means you accept what it indicates. Use the Tab key to move across the choices . . . yes, choose Above

(A) or Below (B), Enter and then S (for Simplify). Derive sees the limit as 9, just as you did.

Try n/(n+1) as n gets bigger and bigger . . . see any pattern to the results? We could substitute 10 for x and calculate and then 100 and then 1000. Use M (for Manage), S (for Substitute) and X for approXimate which turns numbers into decimals. See any trend?

Let's graph y = n/(n+1) and look for a trend visually. If the visual trend and the numerical trend point in the same direction you'll probably be right, at least for a while.

Try $(1+1/n)^n$ as n gets bigger and bigger. Use the same techniques as above. You have to see the problem clearly, $(1+1/5)^5$ or $(1+1/100)^{100}$. Can you predict a result? Guess!! You can always fall back on Derive for correct answers (or a book or a friend or a teacher). If you don't guess and make mistakes you may only be learning how to type.

How I've used Derive with first year algebra students:

Do you have a lot of homework? Is it a real pain? What's it called? . . . ooh, factoring. Let me show you how this computer program can do the problems for you. Do you think you could do those factoring problems without the program? No? Let's see if we can look at this first problem and answer and see if there is a pattern that would allow us to do the problem without the computer.

How I've used Derive with 5th graders:

Do you know what 1*1 is? OK, what about 11*11? Very good! How about 111*111? No? Let's see if Derive can help.

a and b represent numbers. Do you know what a*a*a is? No? Try it on Derive and see. And also try a*b*c*d*a*b*c*d*a. OK? Let me know what happens. Ask Usama if you need help. Or Tim (you know how much he likes to teach).

What are the general uses of Derive in teaching math?

1. To check your answer to a problem that you have solved without Derive.

2. To explore patterns – to allow the student to make generalizations from many instances, more than they could generate without some help.

3. To solve math problems and establish a pattern of success.

4. To start content earlier than usual so that students have lots of time to chew over the ideas and are not rushed.

5. To record students' work. Derive's memory allows me, the teacher, to look at work generated and to think about what was happening. These records are also good reading for the students themselves, for their parents, and for other students as a source of ideas and inspiration.

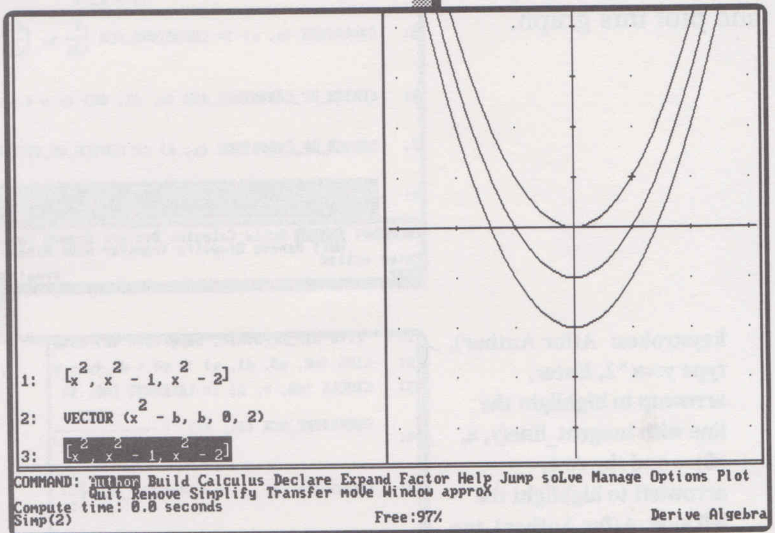
Chapter 16

New Stuff

Many changes have ocurred in Derive in the two and one-half years since the second edition of this book, I've shown some new features in the earlier chapters and in this last chapter I'll show some more.

- Plot will now "distribute" over a list.

  keystrokes: A (for Author), type [x^2, x^2 –1, x^2–2], Enter, W (for Window), S (for Split), V (for Vertical), Enter, F1 (to switch screens), W (for Window), D (for Designate), 2 (for 2D–Plot), Y (for Yes), P.



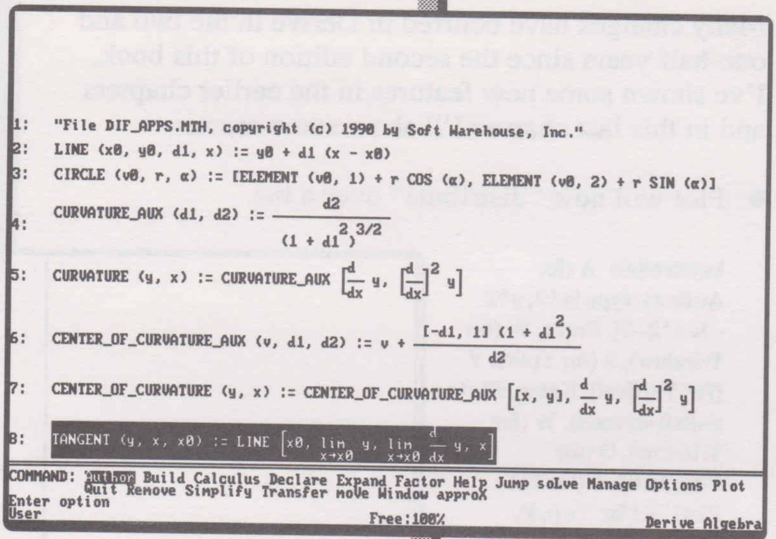
- The vector command will make lists for us:

  keystrokes: A (for Algebra), A (for Author), type vector(x^2–b, b, 0, 2), Enter, S (for Simplify), Enter.

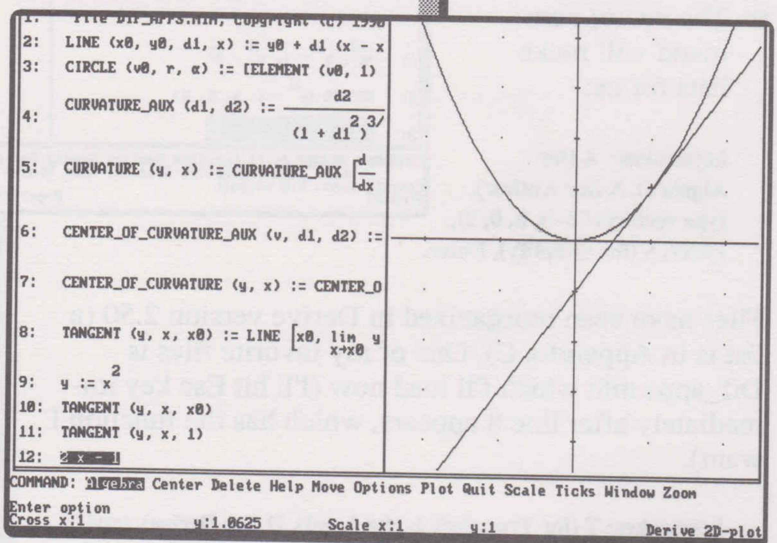
Files have been reorganized in Derive version 2.50 (a list is in Appendix C). One of my favorite files is Dif_apps.mth which I'll load now (I'll hit Esc key immediately after line 8 appears, which has the function I want).

keystrokes: T (for Transfer), L (for Load), D (for Derive), type dif_apps.mth, Enter, type y (for yes), after line 8 appears, hit the Esc key.

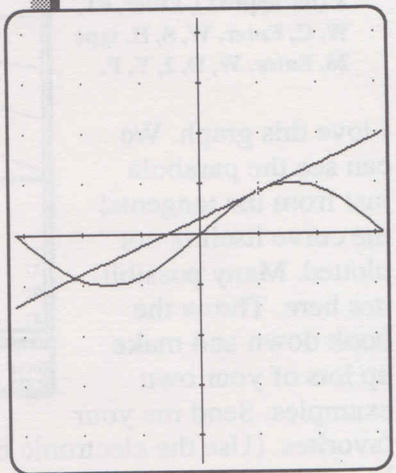
We'll now define a function, make its graph, use the tangent_line function to generate an expression for the tangent to the curve at a specific point, and plot this graph.



keystrokes: A(for Author), type y:=x^2, Enter, arrowup to highlight the line with tangent_line(y, x, x0):= and the rest, arrowleft to highlight the left side, A(for Author), tap key F3 to copy, Enter, M (for Manage), S (for Substitute), Enter three times, tap Del once, type 1, Enter, S (for Simplify), Enter, arrowup to highlight y:=x^2, W (for Window), S (for Split), V (for Vertical), Enter, F1, W, D, 2, Y, P, A (for Algebra), arrowdown to highlight 2x–1, P , P.



We have the graph of a function and the graph of the tangent to that function at a specific location on its graph.



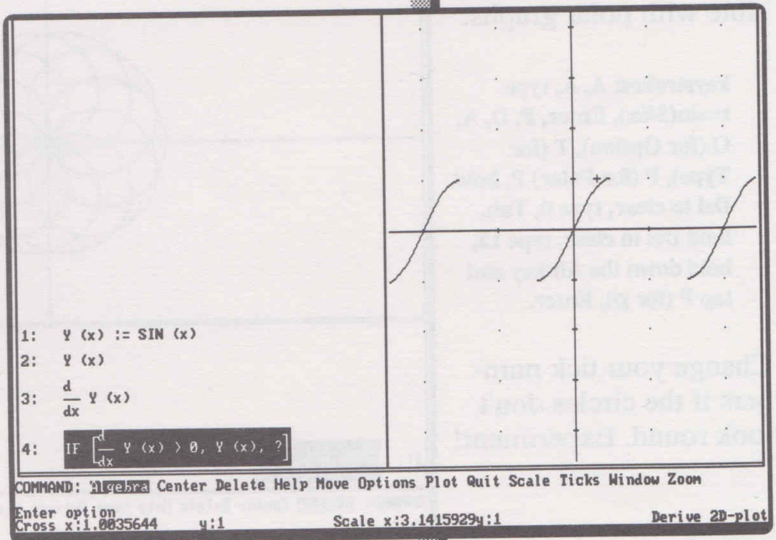
The fun is only beginning. We can change the function, and tangent(y, x, x0) will produce the tangent to this new function.

If two expressions are on a list, Derive thinks we want a parametric graph; by plotting ? as the third element we will get what we want.

keystrokes: A (for Algebra), A (for Author), type y:=sinx, Enter, A (for Author), type [y, tangent(y, x, 1), ?], Enter, S (for Simplify), Enter, P, D, A, P.

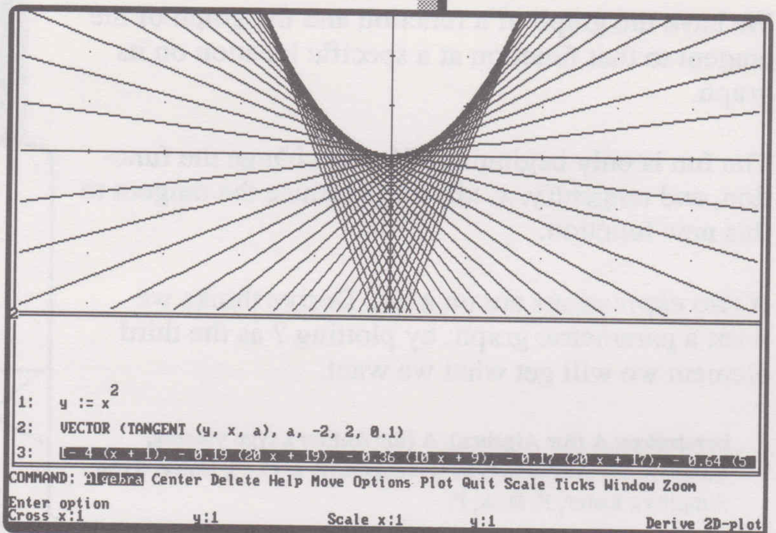
A wonderful idea from Carl Leinbach's book "Calculus Laboratories Using Derive" shows us the use of the if-then-else, structure along with 2-D plotting.

keystrokes: A (for Author), type y(x):=sinx, Enter, A (for Author), type y(x), Enter, C (for Calculus), D (for Differentiate), Enter, Enter, Enter, A (for Author), type if(, F3 (to copy), type >0, y(x), ?), Enter, P (for Plot), D (for Delete), A (for All), P (for Plot).



More is possible! How about a list of expressions that are tangents to a curve, all plotted at the same time in the same window?

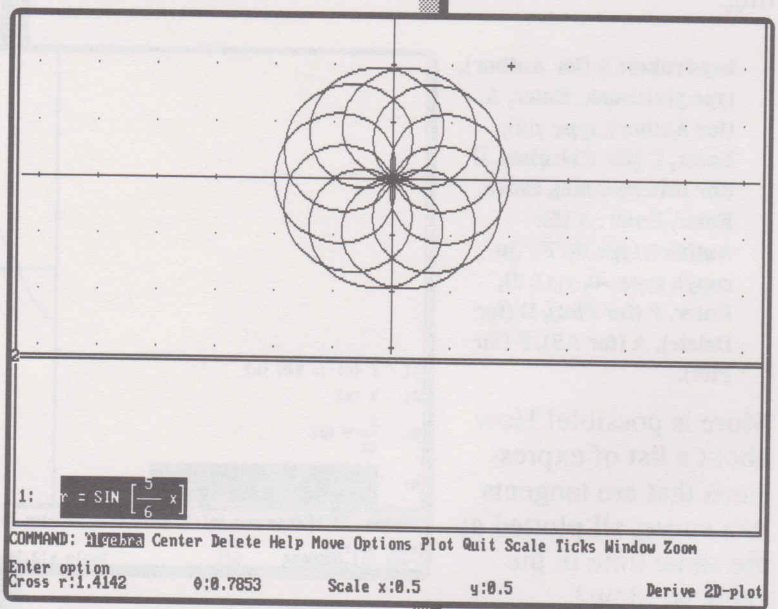
keystrokes: A, A, type y:=x^2, Enter, A, type vector(tangent(y, x, a), a, –2, 2, .1), Enter, X (for approX), Enter, F1, W, C, Enter, W, S, H, type 26, Enter, W, D, 2, Y, P.



I love this graph. We can see the parabola just from the tangents; the curve itself is not plotted. Many possibilites here. Throw the book down and make up lots of your own examples. Send me your favorites. (Use the electronic bulletin board! See the Foreword.)

Nice designs are possible with polar graphs.

keystrokes: A, A, type r=sin(5/6x), Enter, P, D, A, O (for Option), T (for Type), P (for Polar) P, hold Del to clear, type 0, Tab, hold Del to clear, type 12, hold down the Alt key and tap P (for p), Enter.



Change your tick numbers if the circles don't look round. Experiment!

In the past three years we've seen more ways to use the function notation built into Derive. I offer the following development as an opening to finite differences.

keystrokes: A (for Algebra), A (for Author), type f(x):=3x^2, Enter, A (for Author), type vector([x, f(x)], x, 5), Enter, S (for Simplify), Enter.

```
29:  F (x) := 3 x^2
30:  VECTOR ([x, F (x)], x, 5)

     [ 1   3 ]
     [ 2  12 ]
31:  [ 3  27 ]
     [ 4  48 ]
     [ 5  75 ]
```

This gives us a table of values for $f(x) = 3x^2$. I'd like to look at the differences of the f(x)'s, calculate them, and record them for easy viewing.

keystrokes: A (for Author), type vector([x, f(x), f(x+1)–f(x)], x, 5), Enter, S (for Simplify), Enter.

```
32:  VECTOR ([x, F (x), F (x + 1) - F (x)], x, 5)

     [ 1   3   9 ]
     [ 2  12  15 ]
33:  [ 3  27  21 ]
     [ 4  48  27 ]
     [ 5  75  33 ]
```

Now the numbers in our third column are the differences of the values in the second column. Next I'd like the differences in this third column as my fourth column. An easy way of deciding what's needed is to do the third column symbolically. I'll name a new function g and not declare a value for g, and do the differences symbolically.

keystrokes: A (for Author), type g(x):=, Enter, A (for Author), type vector([x, f(x), f(x+1)–f(x), g(x+1)–g(x)], x, 5), Enter, S (for Simplify), Enter.

```
36:  VECTOR ([x, F (x), F (x + 1) - F (x), G (x + 1) - G (x)], x, 5)

     [ 1   3   9  G (2) - G (1) ]
     [ 2  12  15  G (3) - G (2) ]
37:  [ 3  27  21  G (4) - G (3) ]
     [ 4  48  27  G (5) - G (4) ]
     [ 5  75  33  G (6) - G (5) ]
```

Now I can see the fourth column as a symbolic version of the third column. To create my next difference I'll take G(3)–G(2) . . . the second value in the fourth column and subtract (G(2)–G(1)) from it; this produces G(3)–2G(2)+G(1). This tells me what I need for the fourth column: F(x+2)–2F(x+1)+F(1).

keystrokes: A (for Author), type vector([x, f(x), f(x+1)–f(x), f(x+2)– 2f(x+1)+f(x)], x, 5), Enter, S (for Simplify), Enter.

```
38:  VECTOR ([x, F (x), F (x + 1) - F (x), F (x + 2) - 2 F (x + 1) + F (x)], x, 5)
```

$$39:\quad \begin{bmatrix} 1 & 3 & 9 & 6 \\ 2 & 12 & 15 & 6 \\ 3 & 27 & 21 & 6 \\ 4 & 48 & 27 & 6 \\ 5 & 75 & 33 & 6 \end{bmatrix}$$

There must be many other and probably better ways to do this . . . do it.

Now define a function f(x): = $ax^2 + bx + c$ and run the last line of code again and see the lovely result.

We are all warned in the Derive manual, on page 120, that if singularities (blow–ups) occur in an interval, then the integration across this interval may produce incorrect answers. The user is responsible for locating singularities and working around them. The classic example given involves the integral of $1/x^2$ from –1 to 1. The answer demonstrated here will be wrong; after we see this we'll break into two integrals and get it right.

keystrokes: A (for Author), type int(1/x^2, x, –1, 1), Enter, S (for Simplify), Enter.

$$1:\quad \int_{-1}^{1} \frac{1}{x^2}\,dx$$

$$2:\quad -2$$

$$3:\quad \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

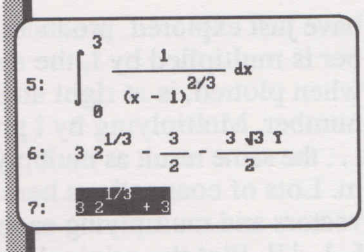
$$4:\quad \infty$$

To get the correct answer:

keystrokes: A (for Author), arrowup, tap key F3 (to copy the highlighted expression), hold down the Ctrl key and tap S three times, tap Del to clear, type 0, hold down the Ctrl key and tap D, type +, tap F3, hold Ctrl and tap S six times, Del, type 0, Enter, S (for Simplify), Enter.

I understand the situation with singularities described above. The following example produces an exact answer, although we integrate right across the blow–up at x=1:

keystrokes: A (for Author), type int(1/(x–1)^(2/3), x, 0, 3), Enter, S (for Simplify), Enter.

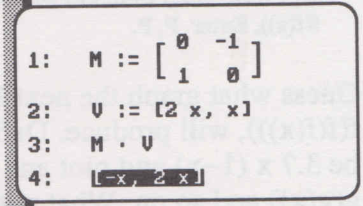


This produces an unpleasant answer with complex number î in it so we fix it:

keystrokes: Arrowup, M (for Manage), B (for Branch), R (for Real), S (for Simplify), Enter.

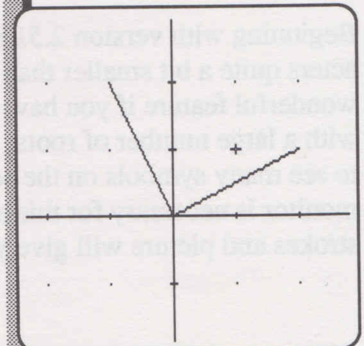
Derive is quite useful with vectors and matrices. A vector, written as [x, 2x], can be plotted and multiplied by a 2 by 2 matrix, and the resulting vector can be plotted. If we want to see the result of more than one multiplication by the matrix we can simply raise the matrix to a power and then multiply the vector; the result is equivalent to a succession of operations by the matrix on the vector and its results.

keystrokes: A (for Author), type m:=[[0,–1], [1,0]], Enter, A (for Author), type v:=[2x, x], Enter, W (for Window), S (for Split), V (for Vertical), Enter, F1, W, D, 2, Y, P, hold Del to clear, type 0, Tab, Del to clear, type 1, Enter.



We see a graph of the vector (2, 1). Now we'll multiply by m (using a decimal point for matrix multiplication) and plot the result and then m^2.v or m.(m.v) and plot the results.

keystrokes: A (for Algebra), A (for Author), type m.v, Enter, S (for Simplify), Enter, P, P, Enter.

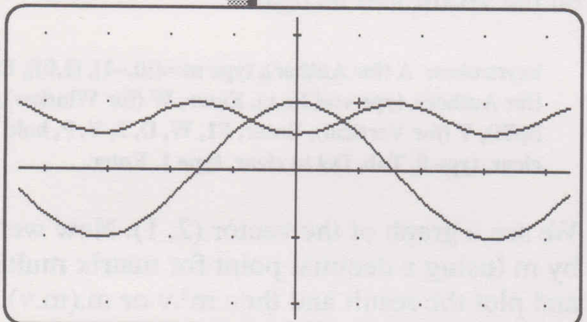


Try m^2.v and m^3.v and so on and see geometrically what results.
A function called Eigenvalues, applied to the m we

have just explored, produces î and –î. If a complex number is multiplied by î, the resulting complex number, when plotted, is at right angles to the plot of the first number. Multiplying by î produces a 90 degree rotation . . . the same result as multiplying a vector by the matrix m. Lots of connections here! Try plotting different vectors and multiplying each by the matrix [[1, –4], [–1, 1]]. Plot the original vector and the vector that results from the matrix multiplication. Can you get the resulting vector to go in the same or exactly opposite direction from the original? Read about eigenvalues and eigenvectors in a book on linear algebra.

Much interest is generated these days by looking at graphs of simple expressions and then looking at recursions of these same expressions.

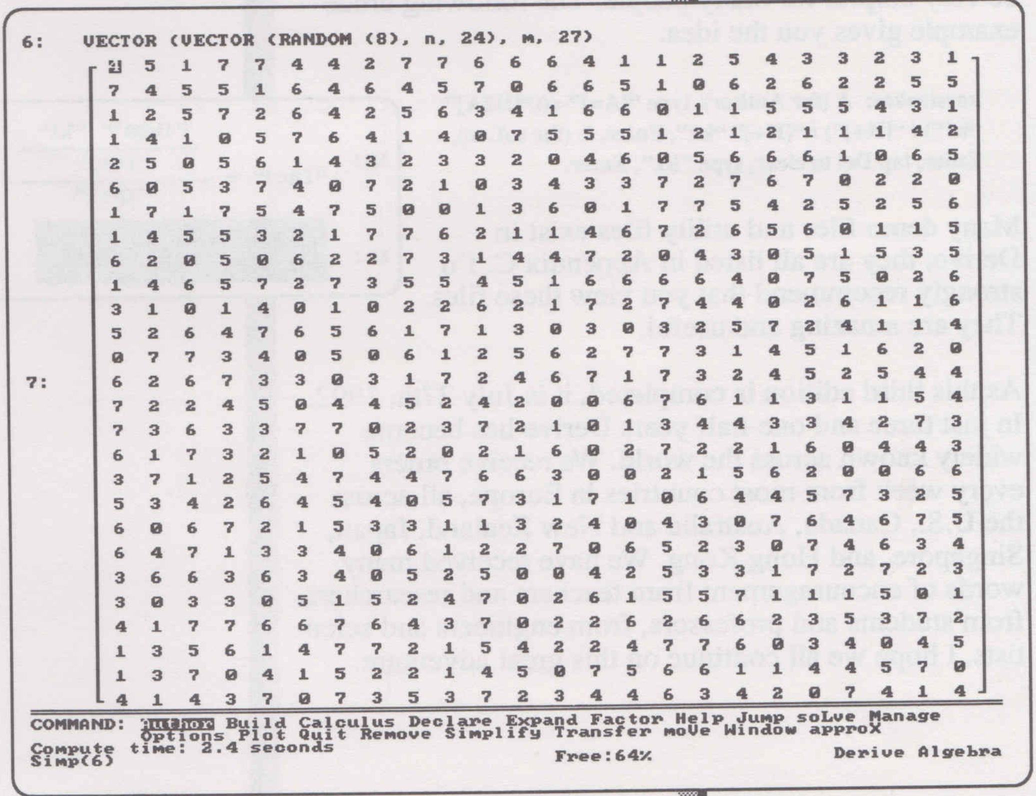
keystrokes: A (for Algebra), A (for Author), type f(x):=cosx, Enter, P, D, A, P, A, A, type f(f(x)), Enter, P, P.



Guess what graph the next level, f(f(f(x))), will produce. Define g(x) to be 3.7 x (1–x) and plot and then g(g(x)) and so on. What patterns do you begin to see? Predict! Change the 3.7 and see the effect.

Beginning with version 2.51 it is possible to generate characters quite a bit smaller than in previous versions. This is a wonderful feature if you have large matrices or equations with a large number of roots; any situation where you need to see many symbols on the screen at one time. A VGA monitor is necessary for this effect. The following keystrokes and picture will give you the idea.

keystrokes: O (for Options), D (for Display), G (for Graphics), H (for High), S (for Small), E (for Extended), V (for VGA), Enter, A (for Author), type vector(vector(random(8), n, 24), m, 27), Enter, S (for Simplify), Enter.



Looking for applications for Derive, I learned how to use a notation that looks a bit like chemistry. The idea should apply in other areas. I want to use a number of symbols in an unusual way; if I place the symbols in quotes I am able to write expressions that normally would cause syntax errors. I can also use symbols like [] and + or – or * without a term following them because these symbols

are buried in parentheses. The amazing fact is that Derive reads each of these piles of symbols in quotes as a variable and all of Derive's capabilities will act on them. We can solve and simplify and expand and factor and plot and substitute as before. I'm sure this idea will be very helpful for many people. The following crude example gives you the idea.

keystrokes: A (for Author), type "[A=]" =(("[H2A]" "k1") / "[H+]") / "[H+]" "k2", Enter, L (for soLve), Enter, tap Del to clear, type "k1", Enter.

10: $$\text{"[A=]"} = \frac{\dfrac{\text{"[H2A]" "k1"}}{\text{"[H+]"}}}{\text{"[H+]"}}\ \text{"k2"}$$

11: $$\text{"k1"} = \frac{\text{"[A=]" "[H+]"}^2}{\text{"[H2A]" "k2"}}$$

Many demo files and utility files exist in Derive; they are all listed in Appendix C. I'd strongly recommend that you view these files. They are amazing and useful.

As this third edition is completed, it is July 27th, 1992. In just three and one-half years Derive has become widely known across the world. We receive orders every week from most countries in Europe, all across the U.S., Canada, Australia and New Zealand, Japan, Singapore, and Hong Kong. We have received many words of encouragement from teachers and researchers, from students and professors, from engineers and scientists. I hope we all continue on this great adventure.

Appendices

Appendix A

Line Editing Commands

While authoring an expression, the text can be edited using the following line edit commands:

Cursor movement commands:

Ctrl S	move left a character
Ctrl D	move right a character
Ctrl A	move left a word
Ctrl F	move right a word
Ctrl Q S	move to left end of line
Ctrl Q D	move to right end of line

Text deletion commands:

Ctrl H	Backspace	delete char left of cursor
Ctrl G	Del	delete char at the cursor
Ctrl T		delete word beginning at cursor
Ctrl Y		delete whole line
Ctrl Q Y		delete right end of line
Ctrl Q H		delete left end of line

Miscellaneous commands:

Ctrl M	Enter	enter line of text
Ctrl J	Ctrl-Enter	enter and simplify line of text
Esc		abort edit and return to menu
Ctrl V	Ins	toggle insert/overwrite modes
Ctrl U		nsert previous line of text
F3		insert highlighted expression
F4		insert highlighted expression enclosed in parentheses

On PC compatible computers, the following Greek letters can be entered by holding down the Alt key while pressing the corresponding Latin (English) letter:

Alt-A	alpha	a	Alt-M	mu	l
Alt-B	beta	b	Alt-P	pi	p
Alt-G	Gamma	C	Alt-S	sigma	r
Alt-D	delta	d	Alt-T	tau	s
Alt-N	epsilon	e	Alt-F	Phi	U
Alt-H	Theta	h	Alt-O	Omega	X

On PC compatible computers, the following mathematical constants and functions can be entered by holding down the Alt key while pressing the corresponding letter:

Alt-E	#e	base of the natural logs
Alt-I	#i	square root of -1
Alt-Q	SQRT	square root function

Functions, Constants, and Operators

Derive knows how to approximate and/or simplify the following mathematical functions, constants, and operators. In approximate mode and given numeric arguments, they are approximated numerically to the current precision level.

Otherwise they are simplified algebraically using powerful transformations. Many transformations are applied automatically; some are applied only when you request them by issuing a Manage command.

Constants:

#e	base of the natural logarithms
#i	square root of -1
pi	circle's circumference to diameter ratio
deg	radians per degree
inf	positive infinity

Operators

- z	minus z
z + w	z plus w
z - w	z minus w
z * w	z times wz w z times w
z / w	z divided by w
z ^ w	z raised to the power w
z %	z percent = z/100
z !	z factorial

Exponential functions:

#e	base of the natural logarithms
EXP (z)	#e raised to the power z
SQRT (z)	square root of z

Logarithmic functions:

LN (z)	natural log of z
LOG (z)	natural log of z
LOG (z, w)	log of z to the base w

Trigonometric functions:

pi	circle's circumference to diameter ratio
deg	radians per degree
SIN (z deg)	sine of z degrees
SIN (z)	sine of z radians
COS (z)	cosine of z radians
TAN (z)	tangent of z radians
COT (z)	cotangent of z radians
SEC (z)	secant of z radians
CSC (z)	cosecant of z radians

Appendix A

Inverse trigonometric functions (radians):

ASIN (z)	angle whose sine is z
ACOS (z)	angle whose cosine is z
ATAN (z)	angle whose tangent is z
ACOT (z)	angle whose cotangent is z
ATAN (y, x)	angle of the point (x,y)
ACOT (x, y)	angle of the point (x,y)
ASEC (z)	angle whose secant is z
ACSC (z)	angle whose cosecant is z

Hyperbolic functions:

SINH (z)	hyperbolic sine of z
COSH (z)	hyperbolic cosine of z
TANH (z)	hyperbolic tangent of z
COTH (z)	hyperbolic cotangent of z
SECH (z)	hyperbolic secant of z
CSCH (z)	hyperbolic cosecant of z

Inverse hyperbolic functions:

ASINH (z)	inverse hyperbolic sine of z
ACOSH (z)	inverse hyperbolic cosine of z
ATANH (z)	inverse hyperbolic tangent of z
ACOTH (z)	inverse hyperbolic cotangent of z
ASECH (z)	inverse hyperbolic secant of z
ACSCH (z)	inverse hyperbolic cosecant of z

Piecewise continuous functions:

ABS (x)	absolute value of z
SIGN (x)	sign of z
MAX (x, y, ...)	maximum of arguments
MIN (x, y, ...)	minimum of arguments
STEP (x)	1 if x>0; 0 if x<0
CHI (a, x, b)	1 if a<x<b; 0 if x<a or x>b

Complex variable functions:

#i	square root of -1
ABS (z)	absolute value of z
SIGN (z)	sign of z
RE (z)	real part of z
IM (z)	imaginary part of z
CONJ (z)	complex conjugate of z
PHASE (z)	phase angle of z

Probability functions:

z!	z factorial
GAMMA (z)	gamma of z
PERM (z, w)	permutations of z things taken w at a time
COMB (z, w)	combinations of z things taken w at a time

Statistical functions:

AVERAGE (z1, ..., zn)	arithmetic mean (average)
RMS (z1, ..., zn)	root mean square
VAR (z1, ..., zn)	variance
STDEV (z1, ..., zn)	standard deviation
FIT (m)	least-squares fit to data matrix m

Error functions:

ERF (z)	error function
ERF (z, w)	generalized error function
ERFC (z)	complementary error function
NORMAL (z, m, s)	normal distribution function with mean m and standard deviation s

Financial functions:

PVAL	(i, nper, pmt, fval, time) present value of contract
FVAL	(i, nper, pmt, pval, time) future value of contract
PMT	(i, nper, pval, fval, time) periodic payment of contract
NPER	(i, pmt, pval, fval, time) number of payment periods of contract

Calculus functions:

LIM (u, x, a) from above	limit of u as x approaches a
LIM (u, x, a, 0) from below	limit of u as x approaches a
DIF (u, x)	derivative of u wrt x
DIF (u, x, n)	nth order derivative of u wrt x
TAYLOR (u, x, a, n)	nth order Taylor approximation of u about a
INT (u, x)	antiderivative of u wrt x
INT (u, x, a, b)	definite integral of u wrt x from a to b
SUM (u, n)	antidifference of u wrt n
SUM (u, n, k, m) from k to m	definite sum of u as n goes
PRODUCT (u, n)	antiquotient of u wrt n
PRODUCT (u, n, k, m)	definite product of u as n goes from k to m

Vector functions:

VECTOR (u, k, m, n, s) a vector of u as k goes from m to n in steps s
ELEMENT (v, n) element n of a vector
v . w dot product of vectors
CROSS (v, w) cross product of two vectors
DIMENSION (v) number of elements of a vector
OUTER (v, w) outer product of two vectors

Matrix functions:

IDENTITY_MATRIX (n) n by n identity matrix
ELEMENT (A, j, k) element in row j and column k of a matrix
A . B dot product of matrices
A' transpose of a matrix (On PC-9801, use Yen char)
DET (A) determinant of a square matrix
TRACE (A) trace (sum of diagonal) of a square matrix
A ^-1 inverse of a square matrix
ROW_REDUCE (A, B) row echelon form of A augmented by B
CHARPOLY (A, x) characteristic polynomial of a square matrix
EIGENVALUES (A) eigenvalues of a square matrix

Differential vector calculus:

GRAD (expn) gradient of expn depending on x, y, and z
GRAD (expn, v) gradient of expn depending on variables in v
GRAD (expn, A) gradient of expn in coordinate system A
DIV (v, A) divergence of a vector
LAPLACIAN (expn, A) divergence of the gradient of expn
CURL (v, A) curl of a vector

Integral vector calculus:

POTENTIAL (v) scalar potential of a vector
POTENTIAL (v, w) potential of a vector with starting coordinates w
VECTOR_POTENTIAL (v) vector potential of a vector
JACOBIAN (v, w) Jacobian matrix of partial derivatives
COVARIANT_METRIC_TENSOR (J) Covariant metric tensor of Jacobian matrix
SQRT_DIAGONAL (G)square roots of the diagonal of a metric tensor

Equation operators and functions:

u = v u equals v
u /= v u not equal v
u < v u less than v
u <= v u less than or equal to v
u > v u greater than v
u >= v u greater than or equal to v

SOLVE (u, x) solve u = 0 for x
SOLVE (u = v, x) solve u = v for x
SOLVE (u = v, x, a, b) solve u = v for x in [a, b]

A Session with Andrea

The work you will see was done by Andrea Monti when she was in the sixth grade. She had begun working on Derive and Mathematica once a week, for a couple of years before this. Andrea's judgment of her math abilities is that she is competent but no genius. Her mother and teachers in school would agree.

I see Andrea as a visual person. On the wall of my workroom are many examples of her math/art creations. She is full of energy and concentration when the work has a visual aspect, especially when she is free to create in a supportive environment.

I never saw Andrea interested in symbolic manipulation. We did some work with equivalent algebraic expressions and factoring polynomial and binomial expansions. Andrea went along politely but you could tell her heart wasn't in it. Then, four months before the session I am about to describe, she arrived for her class at The Math Program and sat by the Mac with Mathematica at the ready. She was ready to give it a try. This started an amazing adventure. I have 40 pages of her work, mostly created by my simple suggestion and then her own creating.

In this session on Derive, I was present only at the beginning and end. We printed out the results with a boldface type to show her input and an italic face to show Derive's response. When she types in a problem, it is her own idea (drawn from previous experiences) and when she types = and an answer it is always done mentally . . . no writing.

On Mathematica, Andrea had typed her name and manipulated it algebraically. On Derive, re is read as a function for the real part of what follows so Derive's response was new to her. See how she struggles with the new conditions.

A friend of mine, Horatio Porta, saw some of Andrea's printouts and said, "Oh, she is teaching herself algebra." I agree and I believe that Andrea's visual sense is driving her algebraic experience.

What you see in these few pages is a tiny window on the marvelous world that will be possible when we put appropriate mathematical tools into the energetic hands of young people. We must also show interest and support their work and provide just a bit of guidance.

999999*999999 = 999998000001

999998000001 = 999998000001

11111111*11111111 = 123456787654321

123456787654321 = 123456787654321

a*n*d*RE(a) *a*n*d*RE(a) *a*n*d*RE(a) *a*n*d*RE(a) *m*o*n*t*i *m*o*n*t*i *m*o*n*t*i *m*o*n*t*i

*a^8*d^4*i^4*m^4*n^8*o^4*t^4*

a*n*d*RE(a) *a*n*d*RE(a) *a*n*d*RE(a) *a*n*d*RE(a) *a*n*d*RE(a) *a*n*d*RE(a) *a*n*d*RE(a) *a*n*d*RE(a) = a*n*d*RE(a)^6

*a^16*d^8*n^8 = a^7*d*n*

a*a*a*a*a*a*a*a*a*a*a*b*b*b*b*b*b*b*b*g*g*g*g*g*t*t*t*t*t = a^11*b^8*g^5*t^5

*a^11*b^8*g^5*t^5 = a^11*b^8*g^5*t^5*

a*n*d*RE(a) *a*n*d*RE(a) *n*d*RE(a)*1^3

*a^5*d^3*n^3*

a*a*a*a*a*a*a*a*a*a*a*a*a*a*a*a*a*a*s*s*s*s = a^18*s*11^4

*a^18*s^4 = 14641*a^18*s*

andrea*andrea*andrea*andrea*andrea*andrea

andrea^6

a*z*a*z*z*za*a*a*a*zz*az = a^4*z^3*az^1*za^1*zz^1

a^5 *az *za *zz *z^3 = a^4 *az *za *zz *z^3

zza *azzz *azzz *azzz *zza *zza *zzza = zza^3 *azzz^3 *zzza^1

azzz^3 *zza^3 *zzza = azzz^3 *zza^3 *zzza

yes *yes *yes *no *no *no *no = yes^3*no^4

no^4*yes^3 = no^4*yes^3

abc*’*s*abc*’*s*abc*’*s*abc*’*s = abc*’*s^4

‘^4*abc^4*s^4 = ‘*abc*s^4

hello *hello *hello *hello *hello *hello *hello = h^7*e^7*l^7*l^7*o^7

hello^7 = e^7*h^7*l^14*o^7

xzc *xcz *xcz *xcz *xcz *xcz *xcz *xcz *xcz *xcz *xdr *xdr *xdr = zcx^10*xdr^3

xcz^9 *xdr^3 *xzc = xdr^3 *zcx^10

h*e*l*l*o*h*e*l*l*o = h^2*e^2*l^2*l^2*o^2

e^2*h^2*l^4*o^2 = e^2*h^2*l^4*o^2

Appendix C

Important Demonstration and Utility Files in Derive

Demonstration Files

Arith.dmo
Algebra.dmo
Trig.dmo
Function.dmo
Calculus.dmo
Matrix.dmo

Utility Files

These files may be loaded into Derive via two different methods:
First method: **keystrokes: Transfer Load Derive filename.mth** - which is slow and all the functions can be seen on the screen.
Second method: **keystrokes: Transfer Load Utility filename.mth** - is much quicker but you will not see the functions on the screen. The functions can be read in the Derive manual.

English.mth	
Metric.mth	
Physical.mth	
Solve.mth	functions to solve systems of non-linear and complex equations.
Vector.mth	many functions to manipulate vectors and matrices...great educational value.
Numeric.mth	functions for numerically approximating derivatives.
Dif_apps.mth	many functions: implicit differentiation, tangent and normal lines to a curve, tangent planes.
Int_apps.mth	functions which generate LaPlace transforms, Fourier series, arc length, area, volume, centroids, areas, and volumes of revolution.
Ode1.mth	
Ode2.mth	
Ode_appr.mth	
Recureqn.mth	
Approx.mth	function to generate Pade rational approximations which can be better than the corresponding Taylor series.
Exp_int.mth	Euler gamma function.
Probabil.mth	Pochhammer, polygamma, Poisson, and binomial functions.
Fresnel.mth	
Bessel.mth	
Hypergeo.mth	
Elliptic.mth	
Orth_pol.mth	Chebychev, Legendre, Hermite, Weber, and Laguerre functions.
Zeta.mth	
Graphics.mth	functions to allow plotting of cylinders, cones, space curves, spheres, and torus.
Misc.mth	many useful functions: ceiling, round, gcd, lcm, Reimann sum approximations, integration by parts, nth prime, Fibonacci, Bernoulli, Catalan, and many others.
Support.pas	

Bibliography

Arney, David C., **Derive Laboratory Manual for Differential Equations**, Addison-Wesley Publishing, Rt. 128, Reading, Massachusetts, 01867. ISBN: 0-201-57268-0.

Cohen, Don, **Calculus By and For Young People (Ages 7, Yes Seven and Up)**, The Math Program, 809 Stratford Dr., Champaign, IL, 61821.

DeMarois, Phil, **College Algebra Laboratories Using Derive**, MathWare, 604 E. Mumford, Urbana, IL., 60801.

Edwards, C.H. Jr., Calculus books of all sorts, Prentice-Hall Publishing

Edwards, C.H. Jr., **The Historical Development of The Calculus**, Springer-Verlag, N.Y., N.Y.

Gilligan, Lawrence G. and Marquadt, James F. Sr.; **Calculus and the Derive Program Second Edition**, Gilmar Publishing Co., ISBN: 0-9626661-2-2.

Leinbach, L. Carl, **Calculus Laboratories Using Derive**, Wadsworth Publishing Co., 10 Davis Dr., Belmont, California, 94002.

Parker, David, **AcroSpin**, Acrobits, P.O. Box 26871, Salt Lake City, Utah, 84126-0871.

Rich, Albert, Rich Joan, Stoutemyer, David, **Derive User Manual**, Soft Warehouse, Inc., 3660 Waialae Ave., Suite 3304, Honolulu, Hawaii, 96816-3236

Sawyer, W.W., **Mathematician's Delight**, Penguin Books; reprinted by Dover Books, N.Y., N.Y.

Appendix E

Production Notes

The layout of the book is a compromise between ease of use and readability. We wanted the reader to be able to easily read and use the book while working with Derive, A Mathematical Assistant (a registered trademark of Soft Warehouse, Inc.) at the computer. Room was needed for recognizable screen shots from Derive sessions to be associated with the keystroke sequences. Also, rather than isolate equations, they were included as part of the text to maintain a conversational style. The trade-off has been some awkward breaks in equations, a result of working in the very large 14pt typeface (before reduction for printing).

The draft of this book was written using Professional Write (a registered trademark of Software Publishing Corp.) on a Toshiba T1000 (a registered trademark of Toshiba America, Inc.) laptop computer. The text was transferred to a 33 mhz 386 computer with VGA monitor, converted to ASCII text with Professional Write and imported into WordPerfect Ver. 5.1 (a registered trademark of WordPerfect Corp.) for clean-up and preparation for placement in PageMaker Ver.4.0 (a registered trademark of Aldus Corp.).

The pictures of graphs and equations were screen dumps taken from a second monochrome graphics monitor attached to the above 33 mhz 386 computer using the Hercules graphics mode in Derive. Pizazz Plus (a registered trademark of Application Techniques, Inc.) was used to make the screen shots and dump them to disk in .TIF format. These shots were then imported directly into PageMaker.

Pictures were resized and reversed (white on black to black on white) from within PageMaker. Equation frames, black backgrounds for graphs and the grey and black border graphics were created using PageMaker's graphics features.

Proof and camera-ready copy was generated on a NEC Silentwriter 2 (Postscript) laser printer with 4 mega-bytes of memory. The pages were reduced to final size with a graphics camera.

The Times New Roman and Symbol typefaces were used..

Mark Deininger
Urbana, IL
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